A note on comprehensive factorization

Claudio Hermida*

15 January 2000

1 Introduction

The purpose of this note is to give a revised account of the comprehensive factorization of a functor. Such factorization was explicitly introduced in [1]. It states that every functor $F: \mathcal{A} \to \mathcal{B}$ can be (uniquely) factored as $F = m \circ e$ where $e: \mathcal{A} \to \mathcal{D}$ is initial and $m: \mathcal{D} \to \mathcal{B}$ is a discrete cofibration. The non-trivial part of the argument is to show $e$ initial. Unfortunately, ibid. fails to give a proof of this. But in fact, the result is correct, and it had been proved earlier by Paré in [2], without acknowledging it as a comprehensive factorization though.

The term 'comprehensive' comes from the fact that we use a comprehension schema to obtain the factorization. We review the general setting for this (a fibration admitting comprehension and sums) in §22. We then go on to review in some detail how the comprehensive factorization works for the simple case of sets and functions (§25), whereby we meet simple 1-dimensional analogues of the constructions involved in the case of functors in §27. There are no original results; we present the background analysis (which is missing in the quoted references) and reorganize the relevant results from [7] to obtain the solution.

2 General setting: comprehension schema on a fibration

Consider a fibration $p: \mathcal{E} \to \mathcal{B}$. We think of it in logical terms: $\mathcal{B}$ is a category of 'sets' and 'functions' and the objects of $\mathcal{E}$ are 'predicates' or 'attributes', while the vertical arrows correspond to 'entailments'. We also assume a 'truth' predicate, which amounts to a fibred terminal object. The concise description is as a right-adjoint-right-inverse $p \dashv \top \mathcal{B}$. In the general setting, comprehension is characterised as a right adjoint $\top \dashv \mathcal{E}$.

In set theory, given a predicate $\phi$ on a set $X$, the comprehension axiom says we can form a set $\{ x : X \mid \phi(x) \}$, the 'extent' of $X$. In the abstract setting, comprehension is characterised as a right adjoint $\top \dashv \mathcal{E}$.

In order to carry out the familiar 'image factorization' of a function, we need to assume 'direct images' in the fibration. These amount to sums, $\Sigma f \dashv f^* \mathcal{E}_J \mathcal{E}_I$ for every $f: I \to J$, satisfying Beck-Chevalley. In the presence of such direct images, we can reformulate the comprehension scheme as follows:

$$\mathcal{B}/J \xrightarrow{\text{Im}} \mathcal{E}_I$$

where $\text{Im}^* / J \mathcal{E}_J$ takes $f: I \to J$ to $\Sigma f(\top_I)$ (its image). This is in fact the original formulation in [2]. The above adjunction gives the desired factorization:

* Dept. of Mathematics and Statistics, McGill University, 805 Sherbrooke St. W., Montreal, QC, Canada H3A 2K6. e-mail: hermida@math.mcgill.ca.
where \( \eta_f \) is the unit of the adjunction. We shall see in the two specific situations below that the construction of the \( \{ \text{Im}(f) \} \) will guarantee automatically its 'monicness' (for the chosen notion of mono). The main work will be to characterise the 'epis' and show \( \eta_f \) is one such. As it happens in the cases below, the 'comprehension' adjunction can usually be decomposed into several ones (essentially two) and the task reduces then to verifying that the respective units are 'epis', and as we shall see there are quite different reasons why they are so.

3 The case: epi-mono factorization of a function

Consider the fibration \( p\text{Fam}(2) \), where \( 2 \) is the usual 2-element \( \{0, 1\} \) Boolean algebra (the subobject classifier in \( ) \). The 'truth' predicate is the functor \( \top\text{Fam}(2) \) with action \( I \mapsto (i: I \mapsto 1) \).

Comprehension has the usual form: given \( \phi I 2 \),

\[
\{ \phi \} = \{ i \in I \mid \phi(i) = 1 \}
\]

therefore a subset of \( I \). It is more conveniently expressed by the following pullback

\[
\begin{array}{ccc}
\{ \phi \} & \to & \{ \cdot \} \\
\downarrow \quad & & \downarrow \phi \\
I & \to & 2
\end{array}
\]

where we see that the left arrow is a mono by construction.

Direct images are described as follows: given \( f I J \) and \( \phi I 2 \),

\[
\Sigma_f(\phi) = (j: J \mapsto \bigvee_{[i \in f^{-1}(j)]} \phi(i))
\]

Now we have the setup to factorize a function \( f I J \) as \( f = m \circ e \) where \( m(\text{Im}(f)) \) is the mono determined by its comprehensive image. What about \( e \)? We expect \( e \) to be an epi in the usual sense. In fact, epis in \( ) \) are simply surjections, which can be expressed logically as

\[
e I Z \text{ epi iff } \Sigma_e(\top_I) = \top_Z
\]

as this effectively means (by the above formula for \( \Sigma_e \)) that the fibers \( e^{-1}(z) \) are non-empty.

2
Let us examine the comprehension adjunction in more detail. First we notice that $2$ is the arrow category $0 \to 1$. We have the (obviously full and faithful) inclusion $i:2$ sending $0$ to $\emptyset$ and $1$ to $\{\bullet\}$. This inclusion has a right adjoint $\pi_2$ with action

$$I \mapsto \bigvee_{i \in I} 1$$

Of course, this boils down to $1$ iff $I \neq \emptyset$, but the above expression is more telling for the purposes of its generalization in §7.7. The unit of this reflection $\eta I \pi_0(I)$ is an epi, since it is the coequalizer

$$I \times I \xrightarrow{\pi} I \xrightarrow{\eta} \pi_0(I)$$

Notice also that $eIZ$ is an epi iff $\forall z: Z. \pi_0(e^{-1}(z)) = 1$.

Back to the comprehension adjunction, we can now express as the composite of adjoints

$$/J \qquad \pi_0 J \qquad 2^J$$

hence the unit $\eta_f \{Im(f)\}$ is the composite

$$I \sim (f^{-1}) \xrightarrow{(\eta f^{-1})} \{\exists(f)\}$$

where the first factor is an isomorphism and the second is epic (since it is epic fibrewise). Hence $\eta_f$ is epic, as we wanted to prove.

**3.1. Remark.** The argument for the epi-mono factorization we have given goes through for any elementary topos $E$. The identification of $E/J$ with $E^J$ requires considering $J$ as an internal discrete category in $E$. Of course, $2$ must be replaced by the subobject classifier $\Omega$.

### 4 The Cat case: the initial-discrete cofibration factorization of a functor

Consider now the fibration $p\text{Fam}(\text{Cat})$, whose fibre at a small category $\mathbb{C}$ is the category of $\mathbb{C}$-valued functors $[\mathbb{C}, \text{Cat}]$, with reindexing given by composition. The ‘truth’ predicate is the functor $\top \text{CatFam}(\text{Cat})$ whose value at $\mathbb{C}$ is the constant functor

$$\top_{\mathbb{C}} = \mathbb{C} \xrightarrow{1} \{\bullet\}$$

The comprehension scheme associates to a functor $P\mathbb{C}$ the corresponding discrete cofibration via the Grothendieck construction. It is useful to exhibit this construction via the following comma square
Direct images amount to left (Kan) extensions: given $F : \mathcal{A} \to \mathcal{B}$ and $P : \mathcal{C} \to \mathcal{B}$ we have

![Diagram](image)

Recall that such a left extension is computed pointwise by the following colimit formula:

$$\Sigma_F(P)(b) = Fb \xrightarrow{P} \mathcal{A} \xrightarrow{P} \mathcal{B}$$

We know now that our 'monics' are going to be the discrete cofibrations. What about the 'epis'? They turn out to be the so called initial functors.

### 4.1 Initial functors and connected components

For a thorough treatment of these concepts see [?].

**4.1. Definition.** A functor $F : \mathcal{A} \to \mathcal{B}$ is **initial** if for every $\Gamma : \mathcal{B} \to \mathcal{C}$,

$$\lim \Gamma F \simeq \lim \Gamma$$

either one existing iff the other one does.

**4.2. Example.** Let $\mathcal{S}$ be a complete lattice. Consider a double-indexed downward chain $\Gamma \omega \times \omega \mathcal{S}$ (i.e. $\Gamma(i, j) \geq \Gamma(i', j')$ whenever $(i, j) \leq (i', j')$). The diagonal $\Delta \omega \times \omega$ is initial, i.e.

$$\bigwedge_{i,j} \Gamma(i, j) = \bigwedge_n \Gamma(n, n)$$

Before proceeding to an explicit (and useful) characterization of initial functors we need to introduce the 'connected components' functor, which will be part of our decomposition of the comprehension adjunction.

The inclusion $\Delta \mathbf{Cat}$ (regarding sets as discrete categories) has a left adjoint $\Pi_0 \mathbf{Cat}$, which takes a category (actually its underlying graph) to the set of connected components. It can be calculated as a colimit:

$$\Pi_0(\mathcal{C}) = \mathcal{C} \longrightarrow 1 \longrightarrow \{\bullet\}$$

Notice that $\Pi_0(\mathcal{C}) = \{\bullet\}$ iff $\mathcal{C}$ is non-empty and pathwise connected.