Fibring Labelled Deduction Systems

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Abstract

We give a categorial characterization of how labelled deduction systems for logics with a propositional basis behave under unconstrained fibring and under fibring that is constrained by symbol sharing. At the semantic level, we introduce a general semantics for our systems and then give a categorial characterization of fibring of models. Based on this, we establish the conditions under which our systems are sound and complete with respect to the general semantics for the corresponding logics, and establish requirements on logics and systems so that completeness is preserved by both forms of fibring.

Keywords: Fibring of Logics, Labelled Deduction Systems, Natural Deduction, General Semantics, Category Theory.

1 Introduction

Context The problem of combining logics has attracted much attention lately. Besides leading to interesting applications whenever it is necessary to work with different logics at the same time, combining logics is also interesting on purely theoretical grounds [7], and among the different techniques for combining logics, *fibring* [15, 17, 18, 25, 26, 29] deserves close study.

The fibring of two logics is obtained by combining their languages, deduction systems and semantics. The language of the fibring is obtained by allowing the use of the language constructors (atomic symbols and connectives) from the logics that we want to combine. So, for example, when fibring a temporal logic and a deontic logic, "mixed" formulae, like $((G\alpha) \Rightarrow (O(F\beta)))$, appear in the resulting logic. In many cases one wants to share some of the symbols, and we then speak of fibring *constrained by symbol sharing*. For example, the formula above illustrates the constrained form of fibring imposed by sharing the common propositional part of a temporal and a deontic logic. Alternatively, we may consider *unconstrained* fibring, where no symbols of the two given logics are shared.

The deduction system of the fibring is obtained by the free use of the inference rules from both given logics, provided that these logics are endowed with deduction systems of the same kind (e.g. both logics with Hilbert systems or both with natural deduction systems). Moreover, this approach is of practical interest only if the two given deduction systems are *schematic*, in the sense that their inference rules are "open" for application to formulae with "foreign" symbols. For instance, when in some Hilbert system *modus ponens* is defined by the rule MP, $\{(\xi_1 \Rightarrow \xi_2), \xi_1\} \vdash \xi_2$, one may implicitly assume that the instantiation of the metavariables ξ_1 and ξ_2 by any formulae, possibly with symbols from both logics, is allowed when applying MP in the fibring.

The semantics of the fibring is complicated and, similar to deduction systems, it is better to consider the case where both logics have semantics of the same kind (i.e. with similar models). A possible, quite general, model for many logics with a propositional basis is provided by a triple $\langle U, \mathcal{B}, \nu \rangle$ where U is a set (of points, worlds, states, whatever), $\mathcal{B} \subseteq \wp U$, and $\nu(c) : \mathcal{B}^n \to \mathcal{B}$ for each language constructor c of arity $n \geq 0$. Given two logics $\mathcal{L}'$ and $\mathcal{L}''$ with such a semantics, the semantics of their fibring is a class of models of the form $\langle U, \mathcal{B}, \nu \rangle$, such that at each point $u \in U$ it is possible to extract a model from $\mathcal{L}'$ and one from $\mathcal{L}''$. If symbols are shared, the two extracted models should agree on them.

This paper follows a program started by [25], where a categorial characterization of fibring of logics with a propositional basis is presented, continued in [26] with the study of fibring of logics with terms and binding operators, and in [29] where preservation of (strong) completeness by fibring is obtained for logics with a propositional basis. All these works consider Hilbert-style deduction systems. Here, we concentrate our attention on fibring of logics endowed with *labelled deduction systems*.

To illustrate why considering these systems is important, observe that Hilbert-style systems, although uniform, are difficult to use in practice, especially in comparison with the more "natural" Gentzen-style systems such as natural deduction, sequent and tableaux systems. Unfortunately, devising Gentzen-style systems for modal, relevance and other non-classical logics often requires considerable ingenuity, as well as trading uniformity for simplicity and usability. A solution to this problem is to employ *labelling* techniques, which provide a general framework for presenting different logics in a uniform way in terms of Gentzen-style deduction systems. The intuition behind these techniques is that labelling (sometimes also called prefixing, annotating or subscripting) allows one to explicitly encode additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. So, for instance, instead of considering a modal formula φ, we can consider the *labelled formula* $w{:}\varphi$, which intuitively means that φ holds at the point denoted by w in U within the underlying semantics. We can also use labels to specify the way in which points are related in a particular model; for example, the *relational formula* $w{:}\mathcal{R}\, w'$ tells us that the world w' is accessible from w in a Kripke model (which is a particular, simple, instance of the general models of the form $\langle U, \mathcal{B}, \nu \rangle$).

Labelled deduction systems have been given for several non-classical logics, e.g. [1, 2, 3, 4, 8, 10, 12, 14, 16, 19, 28]. Research on labelling has focused not only on the design of systems for specific logics, but also, more generally, on the characterization of the classes of logics that can be formalized this way. General properties and limitations of labelling techniques have also been investigated. For example, [4, 28] highlight a tradeoff between limitations and properties: if we reason on the semantic information provided by labelling using Horn-style rules, then we are able to present only a subset of all possible non-classical logics, but we can still capture many of the most common ones and, more importantly, labelling provides an efficient general method for establishing the metatheoretical properties of these logics, including their completeness, decidability and computational complexity. Moreover, [4, 28] also contain some preliminary results on combining labelled deduction systems.

Contributions The main contribution of this paper is an account of how labelled deduction systems can be fibred. We concentrate on logics with a propositional basis, endowed with labelled deduction systems, and where labels provide additional information of a semantic nature. Specifically, our contributions can be summarized as follows. Our approach is general and we give here a novel (algebraic) presentation of labelled deduction systems that provides a suitable basis for defining their fibring, and that subsumes, as simple special cases, the more standard labelled natural deduction systems of our previous work [3, 4, 28]. Based on this presentation, we then give a categorial definition of (unconstrained and constrained by symbol sharing) fibring of labelled deduction systems. Afterwards, we adapt the *general semantics* advocated for Hilbert systems in [29] to the case of labelled deduction systems; hence, the general approach that we consider here subsumes our previous work also at the semantic level (the Kripke-style semantics that we gave for labelled natural deduction systems is a simple, special case of the general semantics). Then we establish the conditions under which our systems are sound and complete with respect to the general semantics for the corresponding logics, and define (both forms of) fibring at the semantic level. Finally, building on the completeness theorem, we establish requirements on the given logics and systems so that completeness is preserved by (both forms of) fibring. Preservation of other metatheoretical properties such as the finite model property and decidability will be subject of future work.

Organization The remainder of this paper is organized as follows. In Section 2 we introduce labelled deduction systems at the abstract level we need and their fibrings. In Section 3 we focus on semantics, and in Section 4 we prove the completeness theorem and the preservation results. In Section 5 we draw conclusions and discuss related and future work.

2 Labelled deduction systems

In this section we introduce our labelled deduction systems for logics with a propositional basis. In Section 2.1 we define the language of our systems, specifying how to build formulae and judgements. Then, leading up to the definition of consequence, in

Section 2.2 we define inference rules and derivations in our systems, giving in passing some examples for modal and temporal logics. Finally, in Section 2.3 we introduce unconstrained and constrained fibring as categorial constructions and illustrate them with an example from modal logic.

2.1 Language

We introduce the notions of signature, formula and judgement.

Definition 1 A *signature* is a pair $\Sigma = \langle C, O \rangle$ of families of language *constructors*, with $C = \{C_k\}_{k \in \mathbb{N}}$ where each C_k is a set, and $O = \{O_k\}_{k \in \mathbb{N}^+}$ where each O_k is a set. The elements of C_k are *formula operators* (or *connectives*) of arity k, and the elements of O_k are *relational operators* of arity k. ■

Notation 2 We fix once and for all a family of *schema variables* $\Xi = \{\Xi_j\}_{j \in \{lb, wf\}}$, where Ξ_{lb} is a set of *label schema variables* and Ξ_{wf} is a set of (well-formed) *formula schema variables*. ■

Formulae in our systems are then built from Σ and Ξ according to the following definition.

Definition 3 The set $L(C, \Xi_{wf})$ of *schema formulae* is defined inductively as follows: (i) $\xi \in L(C, \Xi_{wf})$ for all $\xi \in \Xi_{wf}$; (ii) $c \in L(C, \Xi_{wf})$ for all $c \in C_0$; and (iii) $c(\phi_1, \ldots, \phi_k) \in L(C, \Xi_{wf})$ for all $c \in C_k$ and $\phi_1, \ldots, \phi_k \in L(C, \Xi_{wf})$.

The set $L(O, \Xi_{lb})$ of *relational schema formulae* is defined as the set of $o(w_1, \ldots, w_k)$ for all $o \in O_k$ and $w_1, \ldots, w_k \in \Xi_{lb}$. The set $L(\Sigma, \Xi)$ of *labelled schema formulae* is

$$L(\Sigma, \Xi) = \{w{:}\varphi \mid w \in \Xi_{lb} \text{ and } \varphi \in L(C, \Xi_{wf}) \cup L(O, \Xi_{lb})\}.$$ ■

Notation 4 We write $lb(\Delta)$ to denote the set of all labels (label schema variables) that appear in the formulae of a set $\Delta \subseteq L(\Sigma, \Xi)$ of labelled schema formulae. ■

The examples below, in particular Examples 10 and 11, will illustrate the idea behind these definitions: the connectives are in a many-to-many relationship with the relational operators. This is a generalization of the techniques adopted in [3, 4, 28], where labelled natural deduction systems for several non-classical logics are given by associating each non-classical connective of arity n with a syntactic representative of a semantic relation of arity $n + 1$.[1] To illustrate this, take the simple example of modal logics. We can evaluate formulae built using the unary modal connectives $\Box$ and $\Diamond$ in terms of two distinct binary semantic relations $R_\Box$ and $R_\Diamond$, which we represent in a modal labelled natural deduction system by means of formulae of the form $w{:}\mathcal{R}_\Box\, w'$ and $w{:}\mathcal{R}_\Diamond\, w'$. Moreover, in the case that the logic we are considering is, for example, the basic mono-modal logic K, we have that $R_\Box$ and $R_\Diamond$ coincide so that in our system for that logic the connectives $\Box$ and $\Diamond$ are associated with the same binary relational operator $\mathcal{R}$. Similar observations apply for the other non-classical logics considered in [3, 4, 28], e.g. binary relevant implication is associated with a ternary "compossibility" relation.

[1] Note that the techniques of [3, 4, 28] are in turn an adaptation and extension of the principles of "Boolean algebra with operators" [22] and of "gaggle theory" [13].

In [3, 4, 28] we restricted our attention to relationships between connectives and relational operators that are bijective relationships and where the arities are fixed beforehand. But there are logics in which this is *not* the case; a notable example is given by temporal logics, where the binary "until" connective is associated with a relation that is also binary (cf. Example 11). Hence, here we loosen these restrictions: we allow for a liberal relationship between connectives and relational operators, and we do not fix their arities beforehand.

There is however an aspect in which we follow the restrictions of previous work more closely, and we do so in order to simplify the development. Definition 3 says that in our systems we consider two distinct kinds of unlabelled schema formulae, so that the meaning of a labelled schema formula $w{:}\varphi$ will explicitly depend on φ. If φ is a schema formula $c(\phi_1, \ldots, \phi_k)$, then $w{:}\varphi$ means that the formula $c(\phi_1, \ldots, \phi_k)$ holds at the world denoted by w. If φ is a relational schema formula $o(w_1, \ldots, w_k)$, then $w{:}\varphi$ means that the worlds denoted by $w_1, \ldots, w_k$ are in the relation denoted by o at the world denoted by w. For example, when o is the accessibility relation $\mathcal{R}$ of modal logics, $w_1{:}\mathcal{R}\, w_2$ means that the expression "w_2 is accessible", denoted by $\mathcal{R}\, w_2$, holds at w_1, i.e. that at w_1 we can see w_2.

We do not admit "hybrid" formulae where the operators can be freely mixed, e.g. $c_1(o_1(c_2(w_1, w_2)))$. Formulae of this form are used for instance in presentations of non-classical logics based on *hybrid languages*, e.g. [6], or on *semantic embeddings* (i.e. on translations into first or higher-order logic; see, for example, [23]). Moreover, our relational formulae are atomic and cannot be composed.

A consequence of these restrictions is that we will not be able to give labelled deduction systems for all logics that can be presented using hybrid languages, semantic embeddings or other similar approaches based on labelling or translations. However, like for [3, 4, 28], to which the reader is referred for further details and comparison with related approaches, the restricted language that we adopt will suffice to give labelled deduction systems for several of the most common non-classical logics; in fact, thanks to our generalizations, we will be able to give labelled deduction systems for many more logics than the ones considered in previous work.[2]

Returning to the technical development, we now define what is a judgement. Loosely speaking, a judgement is a relation between a labelled schema formula δ, a finite set of labelled schema formulae Δ, and a finite set of label schema variables Φ, which will allow us to define when a formula follows from a set of formulae (cf. Definitions 21 and 22), and also to represent intermediate steps in a derivation where fresh variables may occur.

Definition 5 A *judgement* $\langle \Delta, \delta, \Phi \rangle$ is a triple where $\Delta \in \wp_{\mathit{fin}} L(\Sigma, \Xi)$ is a finite set of labelled schema formulae, $\delta \in L(\Sigma, \Xi)$ is a labelled schema formula, and $\Phi \subseteq \Xi_{lb}$ is a finite set of label schema variables, which will represent *fresh* variables in a derivation. ■

Notation 6 We write $J(\Sigma, \Xi)$ to denote the set of all judgements. Moreover, we

[2]It is then interesting to investigate if our generalizations allow us to inherit the properties shown for the labelled natural deduction systems of previous work, e.g. normalization of derivations and the subformula property. We conjecture that this is the case, but leave a detailed analysis for future work.

represent a judgement $\zeta = \langle \Delta, \delta, \{w_1, \dots, w_k\} \rangle$ by

$$\langle \Delta, \delta \rangle \ \triangleright w_1, \dots, w_k ,$$

and we also denote Δ by $Ant(\zeta)$, δ by $Cons(\zeta)$, and $\{w_1, \dots, w_k\}$ by $Fresh(\zeta)$. Furthermore, when $Fresh(\zeta)$ is empty, we represent ζ by writing simply $\langle \Delta, \delta \rangle$. ■

2.2 Deduction and consequence

In order to define a consequence relation for proving assertions using hypotheses, which we will do in Prop/Definition 27, we need to say when a judgement follows (i.e. can be derived) from a set of judgements. Hence, we now introduce inference rules and define our labelled deduction systems.

Definition 7 An *inference rule* is a pair $\langle Prem, Conc \rangle$ where $Prem \in \wp_{fin} J(\Sigma, \Xi)$ is a finite set of premise-judgements, and the conclusion-judgement $Conc \in J(\Sigma, \Xi)$ is such that $Ant(Conc) = \emptyset$ and $Fresh(Conc) = \emptyset$. ■

Notation 8 We represent an inference rule $r = \langle \{\zeta_1, \dots, \zeta_k\}, \zeta \rangle$ by

$$\frac{\zeta_1 \quad \cdots \quad \zeta_k}{\zeta}\ r ,$$

and we denote the premises $\{\zeta_1, \dots, \zeta_k\}$ by $Prem(r)$ and the conclusion ζ by $Conc(r)$. We write $R(\Sigma, \Xi)$ to denote the set of all inference rules. ■

Definition 9 A *labelled deduction system* (*lds*) is a pair $\langle \Sigma, D \rangle$ where Σ is a signature and $D \subseteq R(\Sigma, \Xi)$. ■

Even though we imposed some simplifying restrictions on signatures, e.g. that schema formulae cannot be mixed with (atomic) relational schema formulae, our inference rules and lds's are general enough to subsume (as simple instances) the labelled natural deduction systems of [3, 4, 28] for modal, relevance and other non-classical logics.[3]

To illustrate this, we now give lds's for the mono-modal logic K and its extensions, and for basic temporal logic with $\perp$, $\rightarrow$ and the until operator U. In the case of modal logics, we can give lds's that are indeed simple notational variants of the labelled natural deduction systems of previous work, and which contain elimination and introduction rules for each connective c (denoted by cE and cI), except for $\perp$, for which we give only the rule $\perp$E.

[3]By *natural* deduction systems we mean, as is usual, systems for proof under assumption that consist of introduction and elimination rules for each of the connectives except *falsum*, i.e. $\perp$, for which only an elimination rule is given [27]. These rules define the meaning of each connective, specifying how it is introduced in a formula or eliminated from it. Our lds's subsume labelled natural deduction systems because we do not commit ourselves here to such requirements on the rules but allow for the more general form of rules of Definition 7. For example, while we do require that our rules are single-conclusioned, we do not require that they necessarily have the form that makes them introduction or elimination rules. Related to this, it is interesting to investigate whether our general approach generalizes, in a similar way, other forms of labelled Gentzen-style systems such as labelled tableaux or sequent systems; we conjecture that, with minor modifications, this is indeed the case but leave a detailed analysis for future work.

Example 10 An lds $\langle \Sigma, D \rangle$ for the mono-modal logic K is defined as follows: $\Sigma = \langle C, O \rangle$ is such that $C_0 = \Pi \cup \{\bot\}$ where Π is some set of propositional symbols, $C_1 = \{\Box\}$, $C_2 = \{\rightarrow\}$, and $O_1 = \{\mathcal{R}\}$; D contains the rules:

$$\frac{\langle \{w{:}\xi \rightarrow \bot\}, w'{:}\bot \rangle}{\langle \emptyset, w{:}\xi \rangle} \; \bot\mathrm{E}\,,$$

$$\frac{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle \quad \langle \emptyset, w{:}\xi_1 \rangle}{\langle \emptyset, w{:}\xi_2 \rangle} \rightarrow \mathrm{E}\,, \qquad \frac{\langle \{w{:}\xi_1\}, w{:}\xi_2 \rangle}{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle} \rightarrow \mathrm{I}\,,$$

$$\frac{\langle \emptyset, w{:}\Box\xi \rangle \quad \langle \emptyset, w{:}\mathcal{R}\, w' \rangle}{\langle \emptyset, w'{:}\xi \rangle} \; \Box\mathrm{E}\,, \qquad \frac{\langle \{w{:}\mathcal{R}\, w'\}, w'{:}\xi \rangle \;\; \triangleright w'}{\langle \emptyset, w{:}\Box\xi \rangle} \; \Box\mathrm{I}\,.$$

By considering the rule $\Box$I ($\Box$-Introduction), we can now informally explain the role of the finite set of fresh label schema variables. (We will give a formal explanation after Definition 19 below.) Intuitively, the rule $\Box$I says that if for every w' accessible from w we have ξ (i.e. $w'{:}\xi$ follows from $w{:}\mathcal{R}\, w'$ with w' fresh), then we can conclude that in w we have $\Box\xi$ (i.e. $w{:}\Box\xi$). In other words, the universal quantification corresponds to the side condition that w' is fresh, and the antecedent of the premise-judgement requires that w' is accessible from w. ■

Note also that, given C, we can define other formula operators using standard abbreviations, e.g. $\neg\xi \equiv_{abrv} (\xi \rightarrow \bot)$, $\xi_1 \wedge \xi_2 \equiv_{abrv} \neg(\xi_1 \rightarrow \neg\xi_2)$, and $\Diamond\xi \equiv_{abrv} \neg\Box\neg\xi$. The inference rules for these connectives are then derived from those for the primitive ones in C, as we show for instance in Example 24 for $\Diamond$ elimination.

Systems for other modal logics extending mono-modal K are obtained by extending the lds for K with rules expressing properties of $\mathcal{R}$. For example, we obtain lds's for T and K4 by respectively extending the lds for K with the rules

$$\frac{}{\langle \emptyset, w{:}\mathcal{R}\, w \rangle} \; \textit{refl} \qquad \text{and} \qquad \frac{\langle \emptyset, w{:}\mathcal{R}\, w' \rangle \quad \langle \emptyset, w'{:}\mathcal{R}\, w'' \rangle}{\langle \emptyset, w{:}\mathcal{R}\, w'' \rangle} \; \textit{trans}\,,$$

we obtain an lds for S4 by extending the lds for K with both *refl* and *trans*, and an lds for S5 by further adding a rule for the Euclidean property, e.g.

$$\frac{\langle \emptyset, w{:}\mathcal{R}\, w' \rangle \quad \langle \emptyset, w{:}\mathcal{R}\, w'' \rangle}{\langle \emptyset, w'{:}\mathcal{R}\, w'' \rangle} \; \textit{eucl}\,.$$

More generally, we can extend our lds's with *Horn-style rules* similar to the "relational rules" of [3, 4, 28]. That is, with rules of the form

$$\frac{\langle \emptyset, w^1{:}\varphi^1 \rangle \quad \cdots \quad \langle \emptyset, w^n{:}\varphi^n \rangle}{\langle \emptyset, w^0{:}\varphi^0 \rangle}$$

where φ^i is a relational schema formula for each i such that $0 \leq i \leq n$.

Since these example labelled deduction systems are not much different from the ones in [3, 4, 28], we refer the reader to these works for further examples of systems for modal and other non-classical logics, most notably relevance logics.[4] The following

[4]Note that the guarantee that our lds's for K and its extensions are indeed deduction systems for these logics, i.e. that they are sound and complete with respect to the standard Kripke semantics for these logics, follows then trivially by the soundness and completeness of the labelled natural deduction systems of previous work.

example, on the other hand, shows that the general approach taken here allows us to give a labelled (natural) deduction system for basic temporal logic, which was not possible with the formal machinery of previous work.

Example 11 An lds $\langle \Sigma, D\rangle$ for basic temporal logic (with $\bot$, $\rightarrow$, and the until operator U associated with a binary relational operator $\mathcal{R}$) is defined as follows: $\Sigma = \langle C, O\rangle$ is such that $C_0 = \Pi \cup \{\bot\}$ where Π is some set of propositional symbols, $C_2 = \{\rightarrow, U\}$ and $O_1 = \{\mathcal{R}\}$; D contains rules $\bot$E, $\rightarrow$ E, $\rightarrow$ I, *trans*, and the following rules for U:

$$\frac{\langle \emptyset, w{:}\xi_1 U \xi_2\rangle \quad \langle \emptyset, w{:}\mathcal{R}\, w'\rangle \quad \langle \{w{:}\mathcal{R}\, w'', w''{:}\xi_2\}, w'{:}\mathcal{R}\, w''\rangle \;\triangleright w''}{\langle \emptyset, w'{:}\xi_1\rangle}\; U\text{E1}\,,$$

$$\frac{\langle \emptyset, w{:}\xi_1 U \xi_2\rangle \quad \langle \{w{:}\mathcal{R}\, w', w'{:}\xi_2\}, w''{:}\xi_3\rangle \;\triangleright w'}{\langle \emptyset, w''{:}\xi_3\rangle}\; U\text{E2}\,,$$

$$\frac{\langle \emptyset, w''{:}\xi_2\rangle \quad \langle \{w{:}\mathcal{R}\, w', w'{:}\mathcal{R}\, w''\}, w'{:}\xi_1\rangle \;\triangleright w'}{\langle \emptyset, w{:}\xi_1 U \xi_2\rangle}\; U\text{I}\,.$$

■

Note, however, that due to the restrictions we imposed on our language we are not able to give lds's for extensions of this logic where time is linear and/or discrete, which would require us to introduce connectives (e.g. disjunction) with which we can compose relational formulae. Moreover, even though here we can give rules

$$\frac{\zeta_1 \quad \cdots \quad \zeta_k}{\zeta_0}\; r$$

in which we mix judgements ζ_i, where $Cons(\zeta_i)$ is a labelled schema formula, with judgements ζ_j, where $Cons(\zeta_j)$ is a labelled relational schema formula, there are other logics that require further interaction between formula and relational operators. That is, there are logics that require us explicitly to consider hybrid formulae, in which operators can be mixed freely.[5]

We now motivate the following definitions. The objective is to say when a formula γ is derived from a set of formulae Γ in an lds $\langle \Sigma, D\rangle$, i.e. $\Gamma \vdash_{\langle \Sigma, D\rangle} \gamma$. We will say that this is the case when the judgement $\langle \Gamma, \gamma, \emptyset\rangle$ is derived from the empty set of judgements. For this purpose we have to define first what it means that a judgement ζ is derived from a set of judgements Θ in an lds $\langle \Sigma, D\rangle$, i.e. $\Theta \mid\!\sim_{\langle \Sigma, D\rangle} \zeta$, and we do so after introducing some basic notions such as substitutions, instantiations and derivation trees.

Definition 12 A *substitution* is a pair $\sigma = \langle \sigma_{lb}, \sigma_{wf}\rangle$, where $\sigma_{lb} : \Xi_{lb} \rightarrow \Xi_{lb}$ and $\sigma_{wf} : \Xi_{wf} \rightarrow L(C, \Xi_{wf})$ are maps called respectively *label substitution* and *formula substitution.* ■

[5]In other words, we allow mixing of operators at the metalevel, by having rules that mix judgements with different kinds of formulae as conclusions, but we forbid mixing of operators at the object level, i.e. *inside* formulae. Note also that although the inference rules we give here are more general than the ones in [3, 4, 28], the above limitations are a consequence of the results reported in these works as, mutatis mutandis, the observations made there apply straightforwardly in this setting. As we remarked above, labelled deduction systems built using an extended language will be subject of future work.

Notation 13 We write $Sub(\Sigma, \Xi)$ for the set of all substitutions, and *id* for the identity substitution on Ξ. ∎

Since our inference rules are schematic, to be able to carry out derivations we need to define the concepts of substitution for a rule and instantiation. For the former we need to say when two assignments are equivalent up to a particular set of labels.

Definition 14 Let σ and σ' be substitutions and $\Phi \subseteq \Xi_{lb}$ be a set of labels. We say that σ and σ' are Φ*co-equivalent*, in symbols $\sigma \equiv_\Phi \sigma'$, iff $\sigma_{wf} = \sigma'_{wf}$ and $\sigma_{lb}(w) = \sigma'_{lb}(w)$ for each $w \in \Xi_{lb}$ such that $w \notin \Phi$. ∎

Definition 15 Let $r = \langle \{\zeta_i\}_{1\leq i\leq k}, \zeta\rangle$ be an inference rule. An *r-substitution* is a pair $\langle\{\sigma_i\}_{1\leq i\leq k}, \sigma\rangle$ such that $\sigma_i \equiv_{Fresh(\zeta_i)} \sigma$ for each $i = 1, \dots, k$. ∎

Notation 16 In the following, given a labelled schema formula δ, a set of labelled schema formulae Δ and a substitution σ, we denote by $\delta\,\sigma$ the labelled schema formula that results from δ by the simultaneous replacement of its schema variables by their value given by σ. By extension, $\Delta\,\sigma$ denotes the set consisting of all $\gamma\,\sigma$ for each $\gamma \in \Delta$. Similarly, given a label w and a set of labels Φ, we denote by $w\,\sigma$ the label $\sigma_{lb}(w)$, and denote by $\Phi\,\sigma$ the set consisting of all $w\,\sigma$ for each $w \in \Phi$. ∎

Definition 17 The judgement $\langle\Gamma, \gamma, \Upsilon\rangle$ is an *instance* of the judgement $\langle\Delta, \delta, \Phi\rangle$ by the substitution σ iff (i) $\delta\,\sigma$ is γ, (ii) $\Phi\,\sigma \subseteq \Upsilon$, (iii) $\Delta\,\sigma \subseteq \Gamma$, and (iv) $w\,\sigma \notin (\Xi_{lb} \setminus \{w\})\,\sigma$ for each $w \in \Phi$. ∎

Notation 18 We denote by $Inst(\zeta, \sigma)$ the set of instances of a judgement ζ by σ. ∎

Definition 19 The pair $\langle\{\langle\Gamma_i, \gamma_i, \Upsilon_i\rangle\}_{1\leq i\leq k}, \langle\Gamma, \gamma, \Upsilon\rangle\rangle$ is an *instance* of $r = \langle\{\langle\Delta_i, \delta_i, \Phi_i\rangle\}_{1\leq i\leq k}, \langle\emptyset, \delta, \emptyset\rangle\rangle$ by an r-substitution $\langle\{\sigma_i\}_{1\leq i\leq k}, \sigma\rangle$ iff $\delta\,\sigma$ is γ, and for each i, with $1 \leq i \leq k$,

- $\langle\Gamma_i, \gamma_i, \Upsilon_i\rangle \in Inst(\langle\Delta_i, \delta_i, \Phi_i\rangle, \sigma_i)$,
- $lb(\Gamma) \cap \Phi_i\,\sigma_i = \emptyset$,
- $\Gamma_i = \Gamma \cup \Delta_i\,\sigma_i$, and
- $\Upsilon_i \setminus \Phi_i\,\sigma_i = \Upsilon$. ∎

Observe that, reasoning backwards from goal to axioms, i.e. applying rules from conclusion to premises, the image of any fresh variable in a premise should not have been already used in the derivation we are building, and that the hypotheses in the conclusion are transferred to the premises, and so are (more strictly) the fresh variables.

Consider again the rule

$$\frac{\langle\{w{:}\mathcal{R}\,w'\}, w'{:}\xi\rangle \ \triangleright w'}{\langle\emptyset, w{:}\Box\xi\rangle}\ \Box\mathrm{I}\,.$$

The meaning of the "fresh" label schema variable w' can now be specified formally as follows: any instantiation of w' is a variable not present in the formulae of the instance of the judgement except in the ones indicated in the judgement under instantiation (i.e. $w{:}\mathcal{R}\,w'$ and $w'{:}\xi$.)

Notation 20 We denote by $Inst(r, s)$ the set of all rules that are instances of rule r by the r-substitution s. ■

Definition 21 A judgement ζ is *derivable* from a set of judgements Θ in an lds $\langle \Sigma, D\rangle$, in symbols $\Theta \mathrel{|\!\sim}_{\langle \Sigma, D\rangle} \zeta$, iff there is a tree annotated by judgements such that (i) the annotation of the root is ζ, and (ii) for each node,

- the pair formed by the annotation of the node and the set of annotations of its successors is an *instance* of some rule of D,
- or the node has no successors and its annotation is an *axiom*, i.e. it is of the form $\langle \Gamma \cup \{\gamma\}, \gamma, \Upsilon\rangle$,
- or it has no successors and there is an *hypothesis* $\langle \Delta, \delta, \Phi\rangle \in \Theta$ and a substitution $\sigma \equiv_\Phi id$ such that if its annotation has the form $\langle \Gamma, \gamma, \Upsilon\rangle$ then $\Delta\,\sigma \subseteq \Gamma$, $lb(\Delta \cup \{\delta\})\,\sigma \cap \Upsilon \subseteq \Phi\,\sigma$ and γ is $\delta\,\sigma$. ■

A tree in the conditions of the previous definition will be called a *derivation tree* for $\Theta \mathrel{|\!\sim}_{\langle \Sigma, D\rangle} \zeta$. In the following, when there is no ambiguity, we will identify a node with its annotation. When it is not convenient to do so, we will refer to the annotation of a node x by $a(x)$ and even by $a_T(x)$ when referring to the underlying tree T.

We can now finally define what are derivations in our lds's.

Definition 22 A labelled schema formula γ is *derivable* from a set of labelled schema formulae Γ in an lds $\langle \Sigma, D\rangle$, in symbols $\Gamma \vdash_{\langle \Sigma, D\rangle} \gamma$, iff $\emptyset \mathrel{|\!\sim}_{\langle \Sigma, D\rangle} \langle \Gamma, \gamma\rangle$. ■

Notation 23 When no confusion arises, we will simply write $\Theta \mathrel{|\!\sim} \zeta$ and $\Gamma \vdash \gamma$ instead of $\Theta \mathrel{|\!\sim}_{\langle \Sigma, D\rangle} \zeta$ and $\Gamma \vdash_{\langle \Sigma, D\rangle} \gamma$, respectively. ■

As an example, we now employ the derived rules for negation to derive rules for $\Diamond$ in our lds for the mono-modal logic K. All these derived rules are clearly also derivable in lds's extending that for K.

Example 24 It is easy to see that, in our lds for the mono-modal logic K, $\langle \emptyset, w{:}\bot\rangle$ can be derived from $\langle \emptyset, w{:}\neg\xi\rangle$ and $\langle \emptyset, w{:}\xi\rangle$, and that $\langle \emptyset, w{:}\neg\xi\rangle$ can be derived from $\langle \{w{:}\xi\}, w{:}\bot\rangle$. Hence, the following rules for negation are derivable:

$$\frac{\langle \emptyset, w{:}\neg\xi\rangle \quad \langle \emptyset, w{:}\xi\rangle}{\langle \emptyset, w{:}\bot\rangle}\ \neg\mathrm{E} \qquad \text{and} \qquad \frac{\langle \{w{:}\xi\}, w{:}\bot\rangle}{\langle \emptyset, w{:}\neg\xi\rangle}\ \neg\mathrm{I}\,.$$

In (1) we derive $\langle \emptyset, w''{:}\xi_2\rangle$ from the hypotheses $\langle \{w'{:}\xi_1, w{:}\mathcal{R}\, w'\}, w''{:}\xi_2\rangle \,\triangleright w'$ and $\langle \emptyset, w{:}\Diamond\xi_1\rangle$.

$$\cfrac{\cfrac{\cfrac{}{\langle \{w''{:}\xi_2 \to \bot\}, w{:}\neg\Box\neg\xi_1\rangle}\ \mathrm{HYP} \qquad \cfrac{\cfrac{(2)}{\langle \{w''{:}\xi_2 \to \bot, w{:}\mathcal{R}\, w'\}, w'{:}\neg\xi_1\rangle \,\triangleright w'}\ \neg\mathrm{I}}{\langle \{w''{:}\xi_2 \to \bot\}, w{:}\Box\neg\xi_1\rangle}\ \Box\mathrm{I}}{\langle \{w''{:}\xi_2 \to \bot\}, w{:}\bot\rangle}\ \neg\mathrm{E}}{\langle \emptyset, w''{:}\xi_2\rangle}\ \bot\mathrm{E} \tag{1}$$

$$\cfrac{\cfrac{\overline{\langle\{w':\bot\to\bot, w'':\xi_2\to\bot, w':\xi_1, w{:}\mathcal{R}\,w'\}, w'':\xi_2\to\bot\rangle \;\rhd w'}\ \text{AX} \qquad (3)}{\langle\{w':\bot\to\bot, w'':\xi_2\to\bot, w':\xi_1, w{:}\mathcal{R}\,w'\}, w'':\bot\rangle \;\rhd w'}\ \to\text{E}}{\langle\{w'':\xi_2\to\bot, w':\xi_1, w{:}\mathcal{R}\,w'\}, w':\bot\rangle \;\rhd w'}\ \bot\text{E} \tag{2}$$

$$\overline{\langle\{w':\bot\to\bot, w'':\xi_2\to\bot, w':\xi_1, w{:}\mathcal{R}\,w'\}, w'':\xi_2\rangle \;\rhd w'}\ \text{HYP} \tag{3}$$

Hence, the following rule for $\Diamond$ elimination is derivable:

$$\frac{\langle\emptyset, w{:}\Diamond\xi_1\rangle \quad \langle\{w':\xi_1, w{:}\mathcal{R}\,w'\}, w'':\xi_2\rangle \;\rhd w'}{\langle\emptyset, w'':\xi_2\rangle}\ \Diamond\text{E}\,.$$

Similarly, we can derive the introduction rule

$$\frac{\langle\emptyset, w':\xi\rangle \quad \langle\emptyset, w{:}\mathcal{R}\,w'\rangle}{\langle\emptyset, w{:}\Diamond\xi\rangle\rangle}\ \Diamond\text{I}\,,$$

since we can easily show that $\{\langle\emptyset, w':\xi\rangle, \langle\emptyset, w{:}\mathcal{R}\,w'\rangle\} \mid\!\sim \langle\emptyset, w{:}\Diamond\xi\rangle$ ■

Two remarks. First, note that $\Diamond$E inherits from $\Box$I a freshness side condition on a label schema variable: w' is a new variable that is different from w and w'', and does not occur in any formula of the premise-judgement other than $w{:}\mathcal{R}\,w'$ and $w':\xi$. Second, to illustrate the instantiation of rules by given substitutions, note that in the instantiation of $\Box$I (with straightforward substitutions) in the derivation (1) the antecedent of the conclusion ($w'':\xi_2 \to \bot$) is inherited in the premise, and that the other formula ($w{:}\mathcal{R}\,w'$) in the antecedent of the premise is an instantiation of the corresponding formula in the rule.

We now prove two useful properties of derivations, which will allow us to introduce a consequence operator in Prop/Definition 27. We first show that derivations are *monotonic*, in the sense that if a labelled schema formula γ is derivable from a set of labelled schema formulae Γ, then γ can be derived also from $\Gamma \cup \Psi$, where Ψ is an arbitrary set of labelled schema formulae. Then, in Lemma 26, we show that derivations are *idempotent*, since if $\emptyset \mid\!\sim \langle\Gamma, \psi\rangle$ and $\emptyset \mid\!\sim \langle\{\psi\}, \gamma\rangle$ then $\emptyset \mid\!\sim \langle\Gamma, \gamma\rangle$.

Lemma 25 For each $\Psi \subseteq L(\Sigma, \Xi)$ and $\Upsilon \subseteq \Xi_{lb}$ we have $\emptyset \mid\!\sim \langle\Gamma \cup \Psi, \gamma, \Upsilon\rangle$ whenever $\emptyset \mid\!\sim \langle\Gamma, \gamma\rangle$.

Proof Assuming that we are given a derivation tree for $\emptyset \mid\!\sim \langle\Gamma, \gamma\rangle$, we want to define a derivation tree for $\emptyset \mid\!\sim \langle\Gamma \cup \Psi, \gamma, \Upsilon\rangle$. The basic idea is to rename the fresh variables used in the first derivation so that no clash appears with the variables occurring in Ψ and Υ. This is achieved by introducing a family of substitutions that for each node will carry out the renaming of the variables of the judgement in that node.

Consider a derivation tree T for $\emptyset \mid\!\sim \langle\Gamma, \gamma\rangle$, and for each node x define a substitution ρ_x such that

- the substitution of the node root is *id*, and

- if a node x and its successors $\{y_1, \ldots, y_k\}$, where $k \geq 0$, are instances of a rule $r = \langle\{\langle\Delta_i, \delta_i, \Phi_i\rangle\}_{1\leq i\leq k}, \langle\emptyset, \delta, \emptyset\rangle\rangle$ by an r-substitution $\langle\{\sigma_i\}_{1\leq i\leq k}, \sigma\rangle$, then for each $1 \leq i \leq k$ we have that $\rho_{y_i} \equiv_{\Phi_i\,\sigma_i} \rho_x$ and $w\,\sigma_i\,\rho_{y_i} \notin (\Xi_{lb} \setminus \{w\})\,\sigma_i\,\rho_{y_i} \cup lb(Ant(a(x))\,\rho_x \cup \Psi) \cup \Upsilon$ for any $w \in \Phi_i$.

Consider a tree with the same nodes and structure as T and such that each node x has the annotation $\langle\Gamma'\,\rho_x \cup \Psi, \gamma'\,\rho_x, \Upsilon'\,\rho_x \cup \Upsilon\rangle$ if its annotation in T is $\langle\Gamma', \gamma', \Upsilon'\rangle$. We now show that this tree is indeed a derivation tree for $\emptyset \mathrel{|\!\!\sim} \langle\Gamma \cup \Psi, \gamma, \Upsilon\rangle$.

Observe that the annotation of the root is $\langle\Gamma \cup \Psi, \gamma, \Upsilon\rangle$. Observe also that for each node x we have two cases, both of which are straightforward: (i) the node x has no successors and its annotation is an *axiom*, if the same happens in T; and (ii) the pair formed by the annotation of the node x and the annotations of its successors is an *instance* of a rule $r = \langle\{\langle\Delta_i, \delta_i, \Phi_i\rangle\}_{1\leq i\leq k}, \langle\emptyset, \delta, \emptyset\rangle$ by the r-substitution $\langle\{\rho_{y_i} \circ \sigma_i\}_{1\leq i\leq k}, \rho_x \circ \sigma\rangle$, if in T the pair formed by the corresponding annotations is an instance of r by the r-substitution $\langle\{\sigma_i\}_{1\leq i\leq k}, \sigma\rangle$. ∎

Lemma 26 If $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \psi\rangle$ and $\emptyset \mathrel{|\!\!\sim} \langle\{\psi\}, \gamma\rangle$, then $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \gamma\rangle$.

Proof Assume $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \psi\rangle$ and $\emptyset \mathrel{|\!\!\sim} \langle\{\psi\}, \gamma\rangle$. Then, by Lemma 25, there exists a derivation tree T for $\emptyset \mathrel{|\!\!\sim} \langle\Gamma \cup \{\psi\}, \gamma\rangle$. If $\psi \in \Gamma$ then $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \gamma\rangle$. Otherwise let X_ψ be the set of all nodes of T whose annotation is an axiom over ψ and ψ was not introduced by some rule used in the derivation from the root to that node. For each node x in X_ψ, consider a derivation tree T_x for $\emptyset \mathrel{|\!\!\sim} \langle Ant(a(x)) \setminus \{\psi\}, \psi, Fresh(a(x))\rangle$ (observe that T_x exists because $\Gamma \in Ant(\tau(x)) \setminus \{\psi\}$ and $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \psi\rangle$, so we can use Lemma 25). Thus, we can define a derivation tree T' for $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \gamma\rangle$ as follows: the nodes and structure of T' are obtained from T by replacing the nodes x in X_ψ by T_x; for each node x' of T', $a_{T'}(x')$ is

- $a_T(x')$, if x' is a node from T and ψ was introduced in T in the derivation from the root to x';
- $a_{T_x}(x')$, if x' is from T_x, for some $x \in X_\psi$;
- $\langle Ant(a(\overline{x})) \setminus \{\psi\}, Cons(a(\overline{x}))\rangle$, otherwise. ∎

Note that the converse of idempotence of derivations is *interpolation*: if $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \gamma\rangle$ then there exists an interpolant ψ such that $\emptyset \mathrel{|\!\!\sim} \langle\Gamma, \psi\rangle$ and $\emptyset \mathrel{|\!\!\sim} \langle\{\psi\}, \gamma\rangle$. Interpolation holds for some lds's (logics) but may fail for others, so that, unlike the idempotence lemma, we cannot prove such a result for all lds's that fall under our general method.

Prop/Definition 27 The map $\_^{\vdash}\colon \wp(L(\Sigma, \Xi)) \to \wp(L(\Sigma, \Xi))$ such that $\Gamma^{\vdash} = \{\gamma \mid \Gamma \vdash \gamma\}$ is a *consequence operator*.

2.3 Fibring

We now introduce a categorial characterization of fibring of lds's. We begin by defining morphisms between signatures and between lds's. A *signature morphism* $h : \Sigma \to \Sigma'$ is a pair $\langle h^c, h^o\rangle$ of families $h^c = \{h^c_k\}_{k\in\mathbb{N}}$ and $h^o = \{h^o_k\}_{k\in\mathbb{N}^+}$, where each $h^c_k : C_k \to C'_k$ and each $h^o_k : O_k \to O'_k$ are maps. Signatures and signature morphisms constitute the category *Sig*.

We can associate to each signature morphism h a translation map (slightly abusing notation also denoted by h) between the respective languages, judgements and rules. We can then define an *lds morphism* $h : \langle\Sigma, D\rangle \to \langle\Sigma', D'\rangle$ as a morphism $h : \Sigma \to \Sigma'$ in *Sig* such that $h(r) \in D'$ for each $r \in D$.

It is straightforward to prove by induction on the length of derivations that morphisms preserve derivations.

Proposition 28 If $\Theta \mid\!\sim \zeta$ then $h(\Theta) \mid\!\sim' h(\zeta)$, and if $\Gamma \vdash \gamma$ then $h(\Gamma) \vdash' \; h(\gamma)$. ■

Lds's and their morphisms constitute the category *Lds*. Furthermore, the maps $N(\langle\Sigma, D\rangle) = \Sigma$ and $N(h : \langle\Sigma, D\rangle \to \langle\Sigma', D'\rangle) = h$ constitute the forgetful functor $N : \mathit{Lds} \to \mathit{Sig}$. In order to define constrained fibring by sharing constructors (i.e. formula and relational operators) we will employ the construction below, which follows straightforwardly by showing that $\langle\Sigma', h(D)\rangle$ is an lds, that h is an lds morphism, and that the universal property of the cocartesian lifting is fulfilled.

Proposition 29 For each $\langle\Sigma, D\rangle$ in *Lds* and each $h : \Sigma \to \Sigma'$ in *Sig*, the morphism $h : \langle\Sigma, D\rangle \to \langle\Sigma', h(D)\rangle$ is *cocartesian by N for h on $\langle\Sigma, D\rangle$*. ■

We consider two forms of fibring: unconstrained fibring (no sharing of constructors) and constrained fibring (in which constructors are shared). The former is a coproduct, whereas the latter is a cocartesian lifting.

Prop/Definition 30 The *unconstrained fibring* $\langle\Sigma', D'\rangle \oplus \langle\Sigma'', D''\rangle$ of two lds's $\langle\Sigma', D'\rangle$ and $\langle\Sigma'', D''\rangle$ is the coproduct $\langle\Sigma' \oplus \Sigma'', i'(D') \cup i''(D'')\rangle$ endowed with the injections i' and i''. ■

This follows straightforwardly by verifying that $\langle\Sigma' \oplus \Sigma'', i'(D') \cup i''(D'')\rangle$ is an lds, that i' and i'' are morphisms between lds's, and that the universal property of the coproduct is fulfilled.

Definition 31 Let $\langle\Sigma', D'\rangle$ and $\langle\Sigma'', D''\rangle$ be lds's and $f' : \Sigma \to \Sigma'$ and $f'' : \Sigma \to \Sigma''$ injective signature morphisms. The *constrained fibring by sharing* is the codomain of the cocartesian morphism by N for q on $\langle\Sigma', D'\rangle \oplus \langle\Sigma'', D''\rangle$, i.e. $q(\langle\Sigma', D'\rangle \oplus \langle\Sigma'', D''\rangle)$, where q is the coequalizer of $i' \circ f'$ and $i'' \circ f''$. ■

The signature Σ includes the shared constructors, and the morphisms f' and f'' indicate how the shared constructors are related to the constructors in Σ' and Σ''.

To illustrate these definitions, we give an example of fibring lds's for modal logics where all the constructors are shared.

Example 32 Let L' be an lds for K4 with $\Box'$ associated with $\mathcal{R}'$, and L'' be an lds for T with $\Box''$ associated with $\mathcal{R}''$. Fibring of L' and L'' by sharing all constructors (i.e. all operators, including $\Box'$ and $\mathcal{R}'$, and $\Box''$ and $\mathcal{R}''$) yields an lds L with $\Box = \Box' = \Box''$ associated with a relational operator $\mathcal{R}$ that inherits the properties of both $\mathcal{R}'$ and $\mathcal{R}''$, i.e. $\mathcal{R}$ is both transitive and reflexive.

It is easy to verify that L is an lds for S4 (as given above). Rather than proving this, we give an example of a derivation in L, showing in (4) that using the derived $\Diamond$I rule we can derive in L the labelled formula $w{:}\square''\square'\Diamond''\xi \to \Diamond''\square'\square'\square'\Diamond''\xi$, i.e. $w{:}\square\square\Diamond\xi \to \Diamond\square\square\square\Diamond\xi$.

$$\dfrac{\dfrac{\dfrac{(5)}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\mathcal{R}\,w_1\},\, w{:}\square\square\Diamond\xi\rangle \;\rhd w_1}\,\square\mathrm{I}}{\langle\{w{:}\square\square\Diamond\xi\},\, w{:}\square\square\square\Diamond\xi\rangle}\,\square\mathrm{I} \qquad \dfrac{}{\langle\{w{:}\square\square\Diamond\xi\},\, w{:}\mathcal{R}\,w\rangle}\,\mathit{refl}}{\dfrac{\langle\{w{:}\square\square\Diamond\xi\},\, w{:}\Diamond\square\square\square\Diamond\xi\rangle}{\langle\emptyset,\, w{:}\square\square\Diamond\xi \to \Diamond\square\square\square\Diamond\xi\rangle}\,\to\mathrm{I}}\,\Diamond\mathrm{I} \tag{4}$$

$$\dfrac{\dfrac{}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\square w_1,\, w_1{:}\mathcal{R}\,w_2\},\, w{:}\square\square\Diamond\xi\rangle \;\rhd w_1, w_2}\,\mathrm{AX} \qquad (6)}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\mathcal{R}\,w_1,\, w_1{:}\mathcal{R}\,w_2\},\, w_2{:}\square\Diamond\xi\rangle \;\rhd w_1, w_2}\,\square\mathrm{E} \tag{5}$$

$$\dfrac{\dfrac{}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\mathcal{R}\,w_1,\, w_1{:}\mathcal{R}\,w_2\},\, w{:}\mathcal{R}\,w_1\rangle \;\rhd w_1, w_2}\,\mathrm{AX} \qquad (7)}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\mathcal{R}\,w_1,\, w_1{:}\mathcal{R}\,w_2\},\, w{:}\mathcal{R}\,w_2\rangle \;\rhd w_1, w_2}\,\mathit{trans} \tag{6}$$

$$\dfrac{}{\langle\{w{:}\square\square\Diamond\xi,\, w{:}\mathcal{R}\,w_1,\, w_1{:}\mathcal{R}\,w_2\},\, w_1{:}\mathcal{R}\,w_2\rangle \;\rhd w_1, w_2}\,\mathrm{AX} \tag{7}$$

■

What about unconstrained fibring? The unconstrained fibring of L' and L'' yields simply a bi-modal lds with both $\square'$ and $\square''$, where $\mathcal{R}'$ is transitive and $\mathcal{R}''$ is reflexive.

3 Semantics

In this section we introduce interpretation systems building on the work of [29]. We begin Section 3.1 by defining Σ-*structures* and *pre-interpretation systems*, which we use to interpret schema formulae based on assignments for label and formula variables. Relative to pre-interpretation systems, we then define the relevant *entailment operators*. Pre-interpretation systems are however not enough for the purposes of the fibring because we need to have all models that differ only in the names of points (closure under isomorphisms) and whose set of points is the "union" of the sets of points of other models (closure under disjoint unions). In Section 3.2 we thus define *interpretation systems* as pre-interpretation systems closed under isomorphic images and disjoint unions, and in Propositions 51 and 54 we show that such closures do not affect semantic entailment. Moreover, when working with closure under disjoint unions, we can introduce a more standard definition of judgement entailment, which in Proposition 56 we prove to be equivalent to our first definition. We conclude Section 3.2 by defining closure of interpretation systems under subalgebras and showing that also this closure does not affect entailment (Proposition 58). Then in Section 3.3 we focus on fibring at the semantic level and define fibring of interpretation systems both unconstrained and constrained by sharing (Prop/Definitions 62 and 63).

3.1 Satisfaction and entailment

We start by introducing the notion of structure for a signature Σ, where we give a denotation map for each of the operators of Σ.

Definition 33 A Σ-*structure* is a 4-tuple $\langle U, \mathcal{B}, \nu, \mu\rangle$ in which U is a non-empty set, $\mathcal{B} \subseteq \wp U$, $\nu = \{\nu_k\}_{k\in\mathbb{N}}$ with each $\nu_k : C_k \to [\mathcal{B}^k \to \mathcal{B}]$, and $\mu = \{\mu_k\}_{k\in\mathbb{N}^+}$ with each $\mu_k : O_k \to \wp(U^{k+1})$. ■

The elements in U are the *points* of the structure and the elements of $\mathcal{B}$ are its *valuations* (i.e. sets of points in U). To each formula operator c_k we associate via ν_k a map that given a tuple of valuations (those of the arguments of c_k) yields a set of points in U constituting the valuation of the formula. To each o_k we associate via μ_k a $k+1$-relation of elements of U.

Note that we adopt the so-called *general semantics* approach since we consider the possibility of having not all possible valuations but only those in $\mathcal{B}$. (General frames are also used in modal and temporal logics; see, for example, [21].)

Notation 34 We denote the class of all Σ-structures by $Str(\Sigma)$. ■

Definition 35 A *pre-interpretation system* is a 3-tuple $\langle \Sigma, M, A\rangle$ where Σ is a signature, M is a class, and $A : M \to Str(\Sigma)$ is a map that associates a Σ-structure to each $m \in M$. ■

Notation 36 We denote the Σ-*structure* $A(m)$ by $\langle U_m, \mathcal{B}_m, \nu_m, \mu_m\rangle$. ■

To illustrate pre-interpretation systems (and general semantics), take the simple example of modal logic. If Σ_{S4} is the standard signature for S4 (with $\mathcal{R}$ associated to $\Box$), then M is the class of all reflexive and transitive Kripke models. The map A allows us to generalize this by abstracting each Kripke model for S4 to a Σ_{S4}-structure; for instance, if $m = \langle W, R, V\rangle$ is an S4-model, then $A(m) = \langle W, \mathcal{P}(W), \nu, \mu\rangle$ where $\nu_0(p) = V(p)$ for each propositional symbol p, $\nu_0(\bot) = \emptyset$, $\nu_2(\to)(b_1, b_2) = (W\setminus b_1)\cup b_2$, $\nu_1(\Box)(b) = \{u \in W \mid \text{for any } v \in W, \text{ if } uRv \text{ then } v \in b\}$, and $\mu_1(\mathcal{R}) = R$. Mutatis mutandis, further examples can be extracted from the examples given in [29].

Leading up to the definitions of satisfaction and entailment, we now introduce assignments for labels and formulae, and interpretations of schema formulae.

Definition 37 An *assignment* α over a Σ-structure $\langle U, \mathcal{B}, \nu, \mu\rangle$ is a pair $\langle \alpha_{lb}, \alpha_{wf}\rangle$ where $\alpha_{lb} : \Xi_{lb} \to U$ is a *label assignment* for Ξ_{lb} and $\alpha_{wf} : \Xi_{wf} \to \mathcal{B}$ is a *formula assignment* for Ξ_{wf}. ■

When there is no ambiguity, in the sequel we will not refer explicitly to the structure associated with an assignment.

Definition 38 Let $S = \langle U, \mathcal{B}, \nu, \mu\rangle$ be a Σ-structure, α_{wf} a formula assignment for Ξ_{wf}, and α_{lb} a label assignment for Ξ_{lb}. The *interpretation of schema formulae* is a map $[\![\_]\!]^{S,\alpha_{wf}} : L(C, \Xi_{wf}) \to \mathcal{B}$ defined inductively as follows:

- $[\![\xi]\!]^{S,\alpha_{wf}} = \alpha_{wf}(\xi)$ for each $\xi \in \Xi_{wf}$,

- $[\![p]\!]^{S,\alpha_{wf}} = \nu_0(p)$ for each $p \in C_0$, and
- $[\![c(\phi_1, \ldots, \phi_k)]\!]^{S,\alpha_{wf}} = \nu_k(c)([\![\phi_1]\!]^{S,\alpha_{wf}}, \ldots, [\![\phi_k]\!]^{S,\alpha_{wf}})$ for each $k \in \mathbb{N}^+$, $c \in C_k$, and $\phi_1, \ldots, \phi_k \in L(C, \Xi_{wf})$.

The *interpretation of relational schema formulae* is a map

$$[\![\_]\!]^{S,\alpha_{lb}} : L(O, \Xi_{lb}) \to \wp U$$

such that $[\![o(w_1, \ldots, w_k)]\!]^{S,\alpha_{lb}} = \{u \mid (u, \alpha_{lb}(w_1), \ldots, \alpha_{lb}(w_k)) \in \mu_k(o)\}$ for each $k \in \mathbb{N}^+$, $o \in O_k$, and $w_1, \ldots, w_k \in \Xi_{lb}$. ■

Observe that the set $[\![o(w_1, \ldots, w_k)]\!]^{S,\alpha_{lb}}$ is not required to be an element of $\mathcal{B}$. This is a consequence of the definition of the interpretation of relational connectives as k-ary relations on U for some appropriate k.

Definition 39 Let $S = \langle U, \mathcal{B}, \nu, \mu \rangle$ be a Σ-structure, α an assignment, and u a point in U. A schema formula φ is *locally satisfied* by S and α in u, in symbols $S, \alpha, u \Vdash \varphi$, iff $u \in [\![\varphi]\!]^{S,\alpha_{wf}}$ whenever $\varphi \in L(C, \Xi_{wf})$, or $u \in [\![\varphi]\!]^{S,\alpha_{lb}}$ whenever $\varphi \in L(O, \Xi_{lb})$.

A labelled schema formula $w{:}\varphi$ is *locally satisfied* by S and α in u, in symbols $S, \alpha, u \Vdash w{:}\varphi$, iff $S, \alpha, \alpha_{lb}(w) \Vdash \varphi$, and is *globally satisfied* by S and α iff $S, \alpha, u \Vdash w{:}\varphi$ for any $u \in U$. ■

Observe that if we define the interpretation $[\![w{:}\varphi]\!]^{S,\alpha}$ of labelled schema formulae as $\alpha_{lb}(w) \in [\![\varphi]\!]^{S,\alpha}$, then $S, \alpha, u \Vdash w{:}\varphi$ iff $[\![w{:}\varphi]\!]^{S,\alpha}$. More importantly, the previous definition also tells us that a labelled schema formula is satisfied locally iff it is satisfied globally. So, in fact, the satisfaction relations are equivalent. We will represent them simply by $S, \alpha \Vdash \gamma$ since the world plays no role in them, and we will refer to that relation simply as satisfaction relation. Note also that we can extend the satisfaction relation to sets of schema formulae in the obvious way.

The interpretation of fresh variables requires the following notion:

Definition 40 Let α and α' be assignments and $\Phi \subseteq \Xi_{lb}$ be a set of label schema variables. We say that α and α' are Φ*co-equivalent*, in symbols $\alpha \equiv_\Phi \alpha'$, iff $\alpha_{wf} = \alpha'_{wf}$ and $\alpha_{lb}(w) = \alpha'_{lb}(w)$ for each $w \in \Xi_{lb}$ such that $w \notin \Phi$. ■

With this at hand, we can define satisfaction and entailment for judgements in a pre-interpretation system, which will then allow us to define entailment for formulae.

Definition 41 The judgement $\langle \Delta, \delta, \Phi \rangle$ is *satisfied* by a Σ-structure S and an assignment α, in symbols $S, \alpha \Vdash\!\sim \langle \Delta, \delta, \Phi \rangle$, iff for every $\alpha' \equiv_\Phi \alpha$ we have $S, \alpha' \Vdash \delta$ whenever $S, \alpha' \Vdash \Delta$. ■

We can extend the judgement satisfaction relation to sets of judgements in the obvious way. To introduce judgement entailment, we need to define (disjoint) union structures. When considering the union structure of a family of models, we assume that our models have pairwise disjoint sets of points. If this is not the case, we can always make them disjoint by performing a disjoint union.

Definition 42 Let $\langle \Sigma, M, A \rangle$ be a pre-interpretation system and $N = \{n_i\}_{i \in I}$ a non-empty family of models of M. The (*disjoint*) *union structure* S_N of N is the tuple $\langle U, \mathcal{B}, \nu, \mu \rangle$ where $U = \cup_{i \in I} U_{n_i}$; $\mathcal{B} = \{b \subseteq U \mid b \cap U_{n_i} \in \mathcal{B}_{n_i} \text{ for each } i \in I\}$; $\nu_k(c)(b_1, \ldots, b_k) = \cup_{i \in I} \nu_{n_i k}(c)(b_1 \cap U_{n_i}, \ldots, b_k \cap U_{n_i})$; and $\mu_k(o) = \cup_{i \in I} \mu_{n_i k}(o)$. ■

Definition 43 The *judgement entailment operator* for a pre-interpretation system $\langle \Sigma, M, A \rangle$ is a map $\_^{\models\!\sim_p} : \wp J(\Sigma, \Xi) \to \wp J(\Sigma, \Xi)$ such that $\Theta^{\models\!\sim_p} = \{\zeta \mid \text{if } S_N, \alpha \Vdash\!\sim \Theta \text{ then } S_N, \alpha \Vdash\!\sim \zeta \text{ for every non-empty family } N \subseteq M \text{ and } \alpha \text{ over } S_N\}$. ■

We can now introduce entailment for formulae. It is straightforward to show that the entailment operator $\_^{\models_p}$, defined as follows, is a closure operator, i.e. it is extensive, idempotent and monotonic.

Prop/Definition 44 The *entailment operator* for a pre-interpretation system $\langle \Sigma, M, A \rangle$ $\_^{\models_p} : \wp L(\Sigma, \Xi) \to \wp L(\Sigma, \Xi)$, defined as $\Gamma^{\models_p} = \{\gamma \mid \langle \Gamma, \gamma \rangle \in \emptyset^{\models\!\sim_p}\}$, constitutes a closure operator. ■

Notation 45 We represent the fact that $\gamma \in \Gamma^{\models_p}$ by $\Gamma \models_p \gamma$, and use a similar notation for judgement entailment. ■

3.2 Interpretation systems

Following our account above, in order to introduce interpretation systems, we need to define closure under isomorphic structures and closure under disjoint unions. Recall that these closures will be important for the fibring at the semantic level. Indeed, the closure under isomorphic structures, which allows us to abstract away the "names" of the points, is essential for characterizing unconstrained fibring as a coproduct.

Definition 46 A pre-interpretation system $\langle \Sigma, M, A \rangle$ is *closed under isomorphic structures* provided that for every $m \in M$ and bijection $j : U_m \cong U$ there is an $m' \in M$ such that $U_{m'} = U$; $\mathcal{B}_{m'} = \{j^*(b)\}_{b \in \mathcal{B}_m}$, where j^* is the extension of j to sets;

- $\nu_{m'k}(c)(b_1, \ldots, b_k) = j^*(\nu_{mk}(c)(j^{*-1}(b_1), \ldots, j^{*-1}(b_k)))$, for all $k \in \mathbb{N}$, $b_1, \ldots, b_k \in \mathcal{B}_{m'}$, and $c \in C_k$; and
- $\langle w_1, \ldots, w_k \rangle \in \mu_{m'k}(o)$ iff $\langle j^{-1}(w_1), \ldots, j^{-1}(w_k) \rangle \in \mu_{mk}(o)$, for all $k \in \mathbb{N}^+$, and $o \in O_k$. ■

Notation 47 We write $j(A(m))$ for the Σ-structure associated to m'. ■

The closure under isomorphic structures of a pre-interpretation system $\langle \Sigma, M, A \rangle$ is the pre-interpretation system $\langle \Sigma, M', A' \rangle$ where M' is the class of all pairs $\langle m, j \rangle$ with $m \in M$, $j : U_m \to U$ a bijection and $A'(\langle m, j \rangle) = j(A(m))$.

Proposition 51 below shows that closure under isomorphic structures does not affect judgement entailment and, a fortiori, formula entailment. In other words, working with a pre-interpretation system is the same as working with its closure under isomorphic structures. This result is a consequence of the three following lemmas, in which we consider families $N = \{m_i\}_{i \in I}$ of models of M and $N' = \{\langle m'_i, j_i \rangle\}_{i \in I}$ of models of M' where $m_i = m'_i$ for every $i \in I$ (in the sequel we represent $\langle m'_i, j_i \rangle$ by $\langle m_i, j_i \rangle$), and assignments α over S_N and α' over $S_{N'}$ such that $\alpha_{lb}(w) = j_i^{-1}(\alpha'_{lb}(w))$

whenever $\alpha'_{lb}(w) \in U_{\langle m_i, j_i \rangle}$ for any w, and $\alpha_{wf}(\xi) = \cup_{i \in I} j_i^{*-1}(\alpha'_{wf}(\xi) \cap U_{\langle m_i, j_i \rangle})$ for any schema formula ξ.

The first lemma follows straightforwardly by induction on the structure of schema formulae.

Lemma 48 Denotations of schema formulae in S_N and $S_{N'}$ are related as follows: $[\![\phi]\!]^{S_N, \alpha_{wf}} = \cup_{i \in I} j_i^{*-1}([\![\phi]\!]^{S_{N'}, \alpha'_{wf}} \cap U_{\langle m_i, j_i \rangle})$. ■

Lemma 49 Satisfactions of labelled schema formulae in S_N and $S_{N'}$ are related as follows: $S_N, \alpha \Vdash \gamma$ iff $S_{N'}, \alpha' \Vdash' \gamma$. ■

As a proof sketch for this lemma, observe that when γ is $w{:}\varphi$ with φ a schema formula, then we can conclude by applying Lemma 48; when φ is a relational schema formula, the result follows by a straightforward case analysis.

The third lemma relates satisfaction of judgements in a pre-interpretation system and in its closure under isomorphic structures.

Lemma 50 Satisfactions of judgements in S_N and $S_{N'}$ are related as follows: $S_N, \alpha \Vdash\!\sim \langle \Delta, \delta, \Phi \rangle$ iff $S_{N'}, \alpha' \Vdash\!\sim' \langle \Delta, \delta, \Phi \rangle$. ■

This follows by taking into account that for any $\overline{\alpha} \equiv_\Phi \alpha$ there is a $\overline{\alpha}'$ in the conditions of Lemma 49 such that $\overline{\alpha}' \equiv_\Phi \alpha'$, and vice-versa.

Proposition 51 The closure under isomorphic structures of a pre-interpretation system does not affect judgement entailment and entailment. ■

The proof of this proposition is also straightforward, taking into account that for any family of models $\{\langle m'_i, j_i \rangle\}_{i \in I}$ of M' there is a family $\{m'_i\}_{i \in I}$ of models of M satisfying the conditions on models of Lemma 50, and vice-versa, i.e. for any family $\{m_i\}_{i \in I}$ of models of M there is a family $\{\langle m_i, id_i \rangle\}_{i \in I}$ of M' also satisfying those conditions.

Now we define closure under disjoint unions following the same program as for the closure under isomorphic structures (and obtaining similar results, i.e. that the closure does not affect the entailments). The closure under disjoint unions is important because it allows for a more natural definition of entailment as well as a simpler definition of the morphisms between interpretation systems. (The latter fact was already observed in [29].)

Definition 52 A pre-interpretation system $\langle \Sigma, M, A \rangle$ is *closed under disjoint unions* iff for every non-empty family $N = \{n_i\}_{i \in I}$ of models of M, there exists an $n \in M$ such that $A(n)$ is equal to the (disjoint) union structure S_N (cf. Definition 42). ■

The closure under disjoint unions of the pre-interpretation system $\langle \Sigma, M, A \rangle$ is the pre-interpretation system $\langle \Sigma, M', A' \rangle$ where M' is the class of models corresponding to all non-empty families of models of M, and A' is introduced as in Definition 52. Closure under disjoint unions does not affect judgement entailment and, a fortiori, formula entailment. This is a consequence of the following lemma.

Lemma 53 Let $\langle \Sigma, M', A' \rangle$ be the closure under disjoint unions of the pre-interpretation system $\langle \Sigma, M, A \rangle$. Then for each family N' of models of M', there is a family N of models of M such that $S_{N'} = S_N$. ■

As a proof sketch, note that for any family $N' = \{m'_k\}_{k\in K}$ of models of M', where $m'_k = \cup_{i\in I_k} m_{ik}$, we can consider a family $N = \{m_{i\,k}\}_{i\in I_k, k\in K}$ of models of M. Then it is straightforward to show that the components of $S_{N'}$ and S_N are pairwise equal.

Proposition 54 The closure under disjoint unions of a pre-interpretation system does not affect judgement entailment and entailment.

Proof Let $\langle\Sigma, M, A\rangle$ be a pre-interpretation system, and $\langle\Sigma, M', A'\rangle$ its closure under disjoint unions. For judgement entailment, assume $\Theta \mathrel{|\!\!\approx_p} \zeta$ and let $N' = \{m'_k\}_{k\in K}$ be a family of models of M' where $m'_k = \cup_{i\in I_k} m_{ik}$, and α' be an assignment over $S_{N'}$ such that $S_{N'}, \alpha' \mathrel{|\!|\!\!\sim'} \Theta$. Then by Lemma 53 α' is an assignment over S_N where $N = \{m_{i\,k}\}_{i\in I_k, k\in K}$, and so $S_N, \alpha' \mathrel{|\!|\!\!\sim} \Theta$. Therefore $S_N, \alpha' \mathrel{|\!|\!\!\sim} \zeta$ and so $S_{N'}, \alpha' \mathrel{|\!|\!\!\sim'} \zeta$. The other direction follows analogously.

Given this, it is straightforward to show that closure under disjoint unions preserves entailment. ■

Definition 55 A pre-interpretation system is an *interpretation system* iff it is closed under isomorphic images and disjoint unions. ■

We now show that in an interpretation system $\langle\Sigma, M, A\rangle$, judgement entailment $\mathrel{|\!\!\approx_p}$ can be expressed in an equivalent way using a more standard definition where $\_^{\mathrel{|\!\approx}}$ is a map such that

$$\Theta^{\mathrel{|\!\approx}} = \{\zeta \mid \zeta \in \Theta^{\mathrel{|\!\approx}_{A(m)}} \text{ for every } m \in M\}\,,$$

where $\Theta^{\mathrel{|\!\approx}_S} = \{\zeta \mid \text{for every } \alpha \text{ over } S\,,\ \text{if } S, \alpha \mathrel{|\!|\!\!\sim} \Theta \text{ then } S, \alpha \mathrel{|\!|\!\!\sim} \zeta\}$ for a Σ-structure S. In other words, we show that in interpretation systems, checking the entailment for each model independently of the other models in the system is equivalent to checking it in all possible unions of models in the system. This is not the case in pre-interpretation systems, where it may happen that a judgement is satisfied in the union of two models, but it is not satisfied in any of these two models taken singularly. As an example, consider the set of formulae $\{w_1{:}\varphi_1, w_2{:}\varphi_2\}$, two models m, m' and an assignment α over $\{m, m'\}$ such that $\alpha_{lb}(w_1) \in U_m$ and $\alpha_{lb}(w_2) \in U_{m'}$. Assume that $\{m, m'\}, \alpha \Vdash \{w_1{:}\varphi_1, w_2{:}\varphi_2\}$. It is not possible to show in either m or m' satisfaction of both formulae; the same can happen with respect to satisfaction of judgements.

If we had not defined entailment as we did, then we would not have obtained the same entailments when closing under disjoint unions, which is fundamental for fibring. Since for fibring we have to work with interpretations systems, and since we now show that $\mathrel{|\!\!\approx_p}$ and $\mathrel{|\!\approx}$ are equivalent in interpretation systems, we will be able to follow the same program as in [29] (where $\mathrel{|\!\approx}$ is used).

Proposition 56 Let Θ be a set of judgements and ζ a judgement. Then $\Theta \mathrel{|\!\!\approx_p} \zeta$ iff $\Theta \mathrel{|\!\approx} \zeta$.

Proof Left-to-right. Assume $\Theta \mathrel{|\!\!\approx_p} \zeta$. Let $m \in M$ and α be an assignment over m such that $m, \alpha \mathrel{|\!|\!\!\sim} \Theta$. Then $\{m\}$ is a family with a model of M and since $S_m = S_{\{m\}}$ we have $S_{\{m\}}, \alpha \mathrel{|\!|\!\!\sim} \Theta$. Thus $S_{\{m\}}, \alpha \mathrel{|\!|\!\!\sim} \zeta$ and therefore $m, \alpha \mathrel{|\!|\!\!\sim} \zeta$.

Right-to-left. Assume $\Theta \mathrel{|\!\approx} \zeta$. Let N be a family of models of M and α an assignment over S_N such that $S_N, \alpha \mathrel{|\!|\!\!\sim} \Theta$. Then there exists a $n \in M$ such that $A(n) = S_N$, and so $A(n), \alpha \mathrel{|\!|\!\!\sim} \Theta$. Thus $A(n), \alpha \mathrel{|\!|\!\!\sim} \zeta$. ■

For the definition of the fibring, we also need the closure under subalgebras. This property allows us to establish a relationship between interpretation systems (through a morphism) such that for each model of one system there is a model of the other system whose structure is a subalgebra of the structure of the first model.

Definition 57 An interpretation system $\langle \Sigma, M, A \rangle$ is *closed under subalgebras* iff for every $m \in M$ and every ν_m-subalgebra $\mathcal{B}$ of $\mathcal{B}_m$, there is an $m' \in M$ such that $U_{m'} = U_m$; $\mathcal{B}_{m'} = \mathcal{B}$;

- $\nu_{m'k}(c)(b_1, \ldots, b_k) = \nu_{mk}(c)(b_1, \ldots, b_k)$, for all $k \in \mathbb{N}$, $b_1 \ldots, b_k \in \mathcal{B}_{m'}$ and $c \in C_k$; and
- $\mu_{m'k}(o) = \mu_{mk}(o)$, for all $k \in \mathbb{N}^+$ and $o \in O_k$. ■

The closure under subalgebras of an interpretation system $\langle \Sigma, M, A \rangle$ is the interpretation system $\langle \Sigma, M', A' \rangle$ where M' is the class of all pairs $\langle m, \mathcal{B} \rangle$ such that $m \in M$ and $\mathcal{B}$ is a ν_m-subalgebra of $\mathcal{B}_m$ and $A'(\langle m, \mathcal{B} \rangle) = A(m)_{|\mathcal{B}}$.

Proposition 58 The closure under subalgebras of an interpretation system does not affect judgement entailment and entailment.

Proof Let $\langle \Sigma, M, A \rangle$ be an interpretation system, $\langle \Sigma, M', A' \rangle$ its closure under subalgebras, $m' = \langle m, \mathcal{B} \rangle \in M'$, and α an assignment over $A'(m')$. It is straightforward to establish the following properties: (i) $[\![\phi]\!]^{m', \alpha_{wf}} = [\![\phi]\!]^{m, \alpha_{wf}}$; (ii) $m', \alpha \Vdash' \gamma$ iff $m, \alpha \Vdash \gamma$; and (iii) $m', \alpha \Vdash\!\!\sim' \zeta$ iff $m, \alpha \Vdash\!\!\sim \zeta$.

So, assuming $\Theta \models\!\!\approx \zeta$ and $m', \alpha \Vdash\!\!\sim' \Theta$, we have by (iii) that $m, \alpha \Vdash\!\!\sim \Theta$; thus $m, \alpha \Vdash\!\!\sim \zeta$ and therefore $m', \alpha \Vdash\!\!\sim' \zeta$. Assuming now $\Theta \models\!\!\approx' \zeta$, $m, \alpha \Vdash\!\!\sim \Theta$ we have that $\langle m, \mathcal{B} \rangle \in M'$; so $\langle m, \mathcal{B} \rangle, \alpha \Vdash\!\!\sim' \Theta$ and $\langle m, \mathcal{B} \rangle, \alpha \Vdash\!\!\sim' \zeta$, and therefore $m, \alpha \Vdash\!\!\sim \zeta$. ■

From now on, unless otherwise stated, all interpretation systems are assumed to be closed under subalgebras; in other words, we consider only pre-interpretation systems that are closed under isomorphic structures, disjoint unions and subalgebras.

3.3 Fibring

We can now give an account of fibring as a categorial construction of interpretation systems. We begin by defining morphisms between interpretation systems (a notion similar to the one in [29], of course taking now into account labels and relational operators).

Definition 59 An *interpretation system morphism* $h : \langle \Sigma, M, A \rangle \to \langle \Sigma', M', A' \rangle$ is a pair $\langle \overline{h}, \underline{h} \rangle$ where $\overline{h} : \Sigma \to \Sigma'$ and $\underline{h} : M' \to M$ such that for every $m' \in M'$ we have $U_{\underline{h}(m')} = U_{m'}$; $\mathcal{B}_{\underline{h}(m')} = \mathcal{B}_{m'}$;

- $\nu_{\underline{h}(m')k}(c) = \nu_{m'k}(\overline{h}(c))$, for all $k \in \mathbb{N}$ and $c \in C_k$; and
- $\mu_{\underline{h}(m')k}(o) = \mu_{m'k}(\overline{h}(o))$, for all $k \in \mathbb{N}^+$ and $o \in O_k$. ■

Observe that the notion of morphism is very simple (the relationship between models is functional) because of the closure under disjoint unions; if we didn't have this closure, then in each model of the fibring we would have to deal with several models of each component of the fibring. Note also that we can only relate models that have the same set of worlds, but this is not a restriction thanks to the closure under isomorphisms.

In order to show that the fibring of two interpretation systems preserves the entailments of the two systems, we have to show that morphisms preserve entailment.

Proposition 60 Any interpretation system morphism $h : \langle \Sigma, M, A\rangle \to \langle \Sigma', M', A'\rangle$ *preserves judgement entailment* and *entailment*.

Proof For the preservation of judgement entailment, let $m' \in M'$ and α' be a variable assignment over $A'(m')$. Then the following properties hold: (i) $[\![\overline{h}(\phi)]\!]^{m',\alpha'_{wf}} = [\![\phi]\!]^{\underline{h}(m'),\alpha'_{wf}}$; (ii) $m', \alpha' \Vdash' \overline{h}(\gamma)$ iff $\underline{h}(m'), \alpha' \Vdash \gamma$; (iii) $m', \alpha' \Vdash\!\!\sim' \overline{h}(\zeta)$ iff $\underline{h}(m'), \alpha' \Vdash\!\!\sim \zeta$.

So, assuming $\Theta \approx\!\!|\; \zeta$ and $m', \alpha' \Vdash\!\!\sim' \overline{h}(\Theta)$, by (iii) we have $\underline{h}(m'), \alpha' \Vdash\!\!\sim \Theta$, thus $\underline{h}(m')$, $\alpha' \Vdash\!\!\sim \zeta$, and therefore $m', \alpha' \Vdash\!\!\sim' \overline{h}(\zeta)$.

Given this, it is straightforward to show also the preservation of entailment. ■

Interpretation systems and interpretation system morphisms constitute the category *Int*. We follow the same program that we adopted for the labelled deduction systems: we provide an account of unconstrained fibring as a coproduct and constrained fibring as a cocartesian lifting. For this purpose, observe that the maps $N(\langle \Sigma, M, A\rangle) = \Sigma$ and $N(h : \langle \Sigma, M, A\rangle \to \langle \Sigma', M', A'\rangle) = \overline{h}$ constitute the forgetful functor $N : Int \to Sig$.

Proposition 61 For each $\langle \Sigma, M, A\rangle$ in *Int* and each surjective morphism $h : \Sigma \to \Sigma'$ in *Sig*, let the morphism $\hat{h} : \langle \Sigma, M, A\rangle \to \langle \Sigma', M', A'\rangle$ be such that

- M' is the subclass of models m of M such that for each $c', c'' \in C_k$, if $h(c') = h(c'')$ then $\nu_{mk}(c') = \nu_{mk}(c'')$, and for each $o', o'' \in O_k$, if $h(o') = h(o'')$ then $\mu_{mk}(o') = \mu_{mk}(o'')$;
- $A'(m) = \langle U_m, \mathcal{B}_m, \nu', \mu'\rangle$ where $\nu'_k(h_k(c)) = \nu_{mk}(c)$, for all $k \in \mathbb{N}$ and $c \in C_k$, and $\mu'_k(h_k(o)) = \mu_{mk}(o)$, for all $k \in \mathbb{N}^+$ and $o \in O_k$;
- $\hat{h} = \langle h, inc\rangle$.

We then have that $\hat{h}$ is *cocartesian by N for h on $\langle \Sigma, M, A\rangle$*. ■

It is straightforward to verify that $\langle \Sigma', M', A'\rangle = h(\langle \Sigma, M, A\rangle)$ is an interpretation system, that $\hat{h}$ is an interpretation system morphism and that the universal property of the cocartesian lifting holds.

Now we can define both unconstrained and constrained fibring of interpretation systems. Again, the definitions and results are similar to the ones in [29] and we just give proof sketches.

Prop/Definition 62 The *unconstrained fibring* of interpretation systems $\langle \Sigma', M', A'\rangle$ and $\langle \Sigma'', M'', A''\rangle$, in symbols $\langle \Sigma', M', A'\rangle \oplus \langle \Sigma'', M'', A''\rangle$, is the coproduct $\langle \Sigma' \oplus \Sigma'', M, A\rangle$ where

- M is the subclass of $M' \times M''$ composed of the pairs $\langle m', m'' \rangle$ such that $U_{m'} = U_{m''}$ and $\mathcal{B}_{m'} = \mathcal{B}_{m''}$;
- $A(\langle m', m'' \rangle) = \langle U, \mathcal{B}, \nu, \mu \rangle$ where $U = U_{m'}$, $\mathcal{B} = \mathcal{B}_{m'}$ and
  - $\nu_k(i'_k(c')) = \nu_{m'k}(c')$, for all $k \in \mathbb{N}$ and $c' \in C'_k$;
  - $\nu_k(i''_k(c'')) = \nu_{m''k}(c'')$, for all $k \in \mathbb{N}$ and $c'' \in C''_k$;
  - $\mu_k(i'_k(o')) = \mu_{m'k}(o')$, for all $k \in \mathbb{N}^+$ and $o' \in O'_k$;
  - $\mu_k(i''_k(o'')) = \mu_{m''k}(o'')$, for all $k \in \mathbb{N}^+$ and $o'' \in O''_k$;

endowed with the injections $\langle i', p' \rangle$, where i' is the injection $\Sigma' \to \Sigma' \oplus \Sigma''$ and p' is the projection $M' \to M$, and $\langle i'', p'' \rangle$, where i'' is the injection $\Sigma'' \to \Sigma' \oplus \Sigma''$ and p'' is the projection $M'' \to M$. ■

The fact that $\langle \Sigma' \oplus \Sigma'', M, A \rangle$ is an interpretation system and that $\langle i', p' \rangle$ and $\langle i'', p'' \rangle$ are interpretation system morphisms follows straightforwardly. The same applies for the verification of the universal property of the coproduct.

Prop/Definition 63 Let $\langle \Sigma', M', A' \rangle$ and $\langle \Sigma'', M'', A'' \rangle$ be interpretation systems and $f' : \Sigma \to \Sigma'$, $f'' : \Sigma \to \Sigma''$ injective signature morphisms. The *constrained fibring by sharing* of these systems is the codomain of the cocartesian morphism by N for q on $\langle \Sigma', M', A' \rangle \oplus \langle \Sigma'', M'', A'' \rangle$, i.e. $q(\langle \Sigma', M', A' \rangle \oplus \langle \Sigma'', M'', A'' \rangle)$ where q is the coequalizer of $i' \circ f'$ and $i'' \circ f''$. ■

Since coequalizers are epimorphisms in any category, q is surjective in the category of signatures. So we can apply the cocartesian lifting construction.

4 Preservation results

Let a *labelled presentation* (*lp*) of a given logic be a tuple consisting of an lds and an interpretation system for the logic.

When fibring the labelled presentations of two logics it is important to establish which properties of the presentations are preserved. Here we analyze preservation of soundness and completeness. We first characterize the conditions under which a labelled presentation is sound and complete, and then analyze under which conditions soundness and completeness are preserved by fibring.

Given our definitions above, we define the fibring of two lp's $\mathcal{L}' = \langle \Sigma', M', A', D' \rangle$ and $\mathcal{L}'' = \langle \Sigma'', M'', A'', D'' \rangle$ as follows. Their *unconstrained fibring* $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ is the coproduct $\langle \Sigma, M, A, D \rangle$ where $\langle \Sigma, M, A \rangle$ is the unconstrained fibring of interpretation systems $\langle \Sigma', M', A' \rangle$ and $\langle \Sigma'', M'', A'' \rangle$, and $\langle \Sigma, D \rangle$ is the unconstrained fibring of the lds's $\langle \Sigma', D' \rangle$ and $\langle \Sigma'', D'' \rangle$, endowed with the injections $\langle i', p' \rangle : \mathcal{L}' \to \mathcal{L}$ and $\langle i'', p'' \rangle : \mathcal{L}'' \to \mathcal{L}$ such that $\langle i', p' \rangle$ and $\langle i'', p'' \rangle$ are the injections of $\langle \Sigma', M', A' \rangle$ and $\langle \Sigma'', M'', A'' \rangle$ into their coproduct $\langle \Sigma, M, A \rangle$ in *Int*.

Assuming that $f' : \Sigma \to \Sigma'$ and $f'' : \Sigma \to \Sigma''$ are injective signature morphisms, the *constrained fibring by sharing* of $\mathcal{L}'$ and $\mathcal{L}''$ is $q(\mathcal{L}' \oplus \mathcal{L}'')$ where q is the coequalizer of $i' \circ f'$ and $i'' \circ f''$.

Definition 64 An lp $\mathcal{L} = \langle \Sigma, M, A, D \rangle$ is said to be *sound* iff $\Gamma^{\vdash} \subseteq \Gamma^{\models}$ for all $\Gamma \subseteq L(\Sigma, \Xi)$, and it is said to be *complete* iff $\Gamma^{\models} \subseteq \Gamma^{\vdash}$ for all $\Gamma \subseteq L(\Sigma, \Xi)$. ■

Definition 65 Let $\langle \Sigma, M, A, D \rangle$ be an lp. A Σ-structure S is a *structure for* $\langle \Sigma, D \rangle$ iff $Prem(r) \models\!\approx_S Conc(r)$ for all $r \in D$. A model $m \in M$ *is a model for* $\langle \Sigma, D \rangle$ iff $A(m)$ is a structure for $\langle \Sigma, D \rangle$. A rule is *sound* iff any model of M is a model for it. ■

4.1 Conditions for soundness

We first provide the conditions under which an lp is sound, and then analyze the preservation of soundness by both unconstrained and constrained fibring. In order to obtain these results, we prove three lemmas. The first one shows that we can swap the interpretation of a schema formula under a given substitution and a given assignment with the interpretation of the original formula where the substitution is lifted to the assignment. The proof of this lemma is straightforward by induction on the structure of the schema formula.

Lemma 66 Let S be a Σ-structure, α an assignment, σ a substitution and ϕ a schema formula. Then $[\![\phi\, \sigma_{wf}]\!]^{S,\alpha_{wf}} = [\![\phi]\!]^{S,\beta^{\sigma}_{\alpha\, wf}}$ where substitution β^{σ}_{α} is such that $\beta^{\sigma}_{\alpha\, lb}(w) = \alpha_{lb}(\sigma_{lb}(w))$ and $\beta^{\sigma}_{\alpha\, wf}(\xi) = [\![\sigma_{wf}(\xi)]\!]^{S,\alpha_{wf}}$. ■

The second lemma relates the satisfaction of a labelled formula under a given substitution and a given assignment to the satisfaction of the original labelled formula changing the assignment according to the substitution.

Lemma 67 Let S be a Σ-structure, α a variable assignment, σ a substitution and γ a labelled schema formula. Then $S, \alpha \Vdash \gamma\, \sigma$ iff $S, \beta^{\sigma}_{\alpha} \Vdash \gamma$ where β^{σ}_{α} is defined as in Lemma 66. ■

This lemma follows straightforwardly by considering two cases, depending on whether or not φ in $w{:}\varphi$ is a relational schema formula.

The third lemma is important to show that in a derivation the inferences based on sound rules are sound. The proof, although technical, follows straightforwardly by a case analysis.

Lemma 68 Every instance of a sound rule is also sound. ■

By induction on the length of the derivation tree for $\emptyset \mid\!\sim \langle \Gamma, \gamma \rangle$, we can easily prove that $\emptyset \models\!\approx \langle \Gamma, \gamma \rangle$ whenever $\emptyset \mid\!\sim \langle \Gamma, \gamma \rangle$. This implies:

Theorem 69 If every model in an lp $\mathcal{L} = \langle \Sigma, M, A, D \rangle$ is a model for $\langle \Sigma, D \rangle$, then $\mathcal{L}$ is sound. ■

Preservation of soundness follows straightforwardly by exploiting the fact that interpretation systems morphisms preserve entailment (Proposition 60). That is, if we fibre two sound lp's, then we obtain an lp that is also sound.

Theorem 70 The unconstrained and constrained fibring of labelled presentations preserve soundness. ■

4.2 Conditions for completeness

We first provide the conditions under which a labelled presentation of a logic is complete, focusing our attention to this end on lp's that are full and in which we have congruence for labelled formulae. Then, we provide the conditions under which completeness is preserved by (both forms of) fibring.

Definition 71 An lp $\langle \Sigma, M, A, D \rangle$ is said to be *full* iff the image of A is the class of all structures for $\langle \Sigma, D \rangle$. ■

Definition 72 An lds $\langle \Sigma, D \rangle$ is said to be an *lds with congruence* iff the following metatheorem of congruence holds, where $\Gamma \subseteq L(\Sigma, \Xi)$, $k > 0$, $c \in C_k$, $w_3 \in \Xi_{lb}$, and $\phi_1, \ldots, \phi_k, \phi'_1, \ldots, \phi'_k \in L(C, \Xi_{wf})$. If for each $i = 1, \ldots, k$ we have that both

$$\Gamma \cup \{w_1{:}\phi_i\} \vdash w_1{:}\phi'_i \text{ for every } w_1$$

and

$$\Gamma \cup \{w_2{:}\phi'_i\} \vdash w_2{:}\phi_i \text{ for every } w_2,$$

then we also have that

$$\Gamma \cup \{w_3{:}c(\phi_1, \ldots, \phi_k)\} \vdash w_3{:}c(\phi'_1, \ldots, \phi'_k).$$

An lp $\langle \Sigma, M, A, D \rangle$ is an *lp with congruence* iff $\langle \Sigma, D \rangle$ is an lds with congruence. ■

Observe that congruence does not apply to relational schema formulae since we forbid nested constructions.

Completeness is proved by showing that if a formula γ is not a consequence of a set of formulae Γ, then γ is not entailed by Γ. This means that we have to find a model that satisfies all the formulae in the set Γ but does not satisfy the formula γ. To build such a model, we can adopt a *Henkin-style construction* to build a *canonical structure* that provides a counter-model to an unprovable formula. The canonical structure is obtained from a maximal set of formulae, and so we adapt to labelled formulae the standard definition of maximality of a set of formulae with respect to a formula: we say that a set Γ of labelled formulae is *maximal* with respect to a labelled formula γ iff (i) $\Gamma \nvdash \gamma$ and (ii) for any Γ' such that $\Gamma \subset \Gamma'$ we have $\Gamma' \vdash \gamma$.

Recall that here we are not proving completeness of all our lp's but we are providing the general conditions under which an lp is complete. Since the details of the Henkin-style construction depend on the particular logic we are considering, which might be a common modal logic or some more "obscure" non-classical logic, here we can only illustrate the main ideas underlying the construction. To this end, note first that the signature of the lp that we are considering might not contain $\bot$ (indeed, we might be considering a purely positive logic in which negation is not present at all). Hence, we will not define maximality of Γ by relying on its consistency, i.e. that $w{:}\bot$ does not follow from Γ, but rather build Γ so that it is maximal *with respect to* γ, i.e. it satisfies the conditions (i) and (ii) above.

Assume that we are given a labelled schema formula γ and a set Γ_0 of labelled schema formulae such that γ is not a consequence of Γ_0. To extend Γ_0 to a set Γ maximal with respect to γ, let us extend our language with infinitely many new label schema variables (which will act as *witnesses* to the truth of some of the formulae in our language), and let us consider an enumeration of all labelled schema formulae $\gamma_0, \gamma_1, \gamma_2, \ldots$, including the formulae that we can build with these new labels. We then build a sequence of sets $\Gamma_0, \Gamma_1, \ldots$ as follows.

If γ follows from $\Gamma_i \cup \{\gamma_i\}$ using the rules of the lds we are considering, then we simply define $\Gamma_{i+1} = \Gamma_i$. If γ does not follow from $\Gamma_i \cup \{\gamma_i\}$, then we distinguish two cases depending on γ_i.

If $\gamma_i = w_i{:}\phi_i$ and the outermost operator in ϕ_i is an "existential" non-classical connective (e.g. modal $\Diamond$) or the negation of a "universal" non-classical connective (e.g. when ϕ_i is $\neg\Box\phi'$ or $\neg(\phi' \Rightarrow \phi'')$ for relevant implication $\Rightarrow$), then we must also add formulae that witness the truth of $w_i{:}\phi_i$.[6] For example, when γ does not follow from $\Gamma_i \cup \{w_i{:}\Diamond A\}$ in a modal lds, then we pick a new label constant $w_j \notin lb(\Gamma_i \cup \{\gamma_i\})$ to act as a *witness world* to the truth of $w_i{:}\Diamond A$, and define $\Gamma_{i+1} = \Gamma_i \cup \{w_i{:}\Diamond A, w_j{:}A, w_i \mathcal{R}\, w_j\}$. More complex forms of operators are possible, such as the until operator U considered in Example 11, which is an existential connective relatively to position 2. Hence, more generally, if γ_i is $w_i{:}c(\delta_1, \ldots, \delta_k)$ where $c \in C_k$ is an existential operator with respect to positions $j_1, \ldots, j_n$, with $n \leq k$, and o_c is the relational operator related to c, then we define $\Gamma_{i+1} = \Gamma_i \cup \{w_i{:}c(\delta_1, \ldots, \delta_k), w_{j_1}{:}\delta_{j_1}, \ldots, w_{j_n}{:}\delta_{j_n}, w_i{:}o(w_{j_1}, \ldots, w_{j_n})\}$ for the new label constants $w_{j_1}, \ldots, w_{j_n} \notin lb(\Gamma_i \cup \{\gamma_i\})$.

If $\gamma_i = w_i{:}\phi_i$ and the outermost operator in ϕ_i is neither an existential non-classical connective nor the negation of a universal non-classical connective, then we simply define $\Gamma_{i+1} = \Gamma_i \cup \{w_i{:}\phi_i\}$.

The set Γ maximal with respect to γ is then obtained by defining $\Gamma = \cup_{i \geq 0} \Gamma_i$. More detailed examples and discussions can be found in [4, 28]. Indeed, the construction given there for labelled natural deduction systems for modal, relevance and other non-classical logics is a special case of the general construction we have just given. This illustrates once again how our work here subsumes not only the systems of our previous work, but also the metatheoretical results obtained there.

Notation 73 In the sequel we denote $\{w \mid w{:}\varphi \in \Gamma\}$ by $\mathrm{Lab}_\Gamma(\varphi)$, or by $\mathrm{Lab}(\varphi)$ when there is no risk of confusion. ■

Lemma 74 Let the set of labelled formulae Γ be maximal with respect to a labelled formula γ. Then, for any $\varphi, \varphi' \in L(C, \Xi_{wf}) \cup L(O, \Xi_{lb})$, $\mathrm{Lab}(\varphi) = \mathrm{Lab}(\varphi')$ iff $\Gamma \cup \{w_1{:}\varphi\} \vdash w_1{:}\varphi'$ and $\Gamma \cup \{w_2{:}\varphi'\} \vdash w_2{:}\varphi$ for every $w_1, w_2 \in \Xi_{lb}$.

Proof For the left-to-right direction (just for the first derivation since the other is similar), assume $\mathrm{Lab}(\varphi) = \mathrm{Lab}(\varphi')$. There are two cases to consider. If $w_1{:}\varphi \in \Gamma$, then $w_1{:}\varphi' \in \Gamma$ and so, by extensivity, $\Gamma \cup \{w_1{:}\varphi\} \vdash w_1{:}\varphi'$. If $w_1{:}\varphi \notin \Gamma$, then since Γ is maximal we have $\Gamma \cup \{w_1{:}\varphi\} \vdash w_1{:}\varphi'$.

For the right-to-left direction, assume that $\Gamma \cup \{w_1{:}\varphi\} \vdash w_1{:}\varphi'$ and $\Gamma \cup \{w_2{:}\varphi'\} \vdash w_2{:}\varphi$ for every $w_1, w_2 \in \Xi_{lb}$, and $w \in \mathrm{Lab}(\varphi)$. Then $w{:}\varphi \in \Gamma$ and so $\Gamma \vdash w{:}\varphi'$. Therefore $w{:}\varphi' \in \Gamma$ since Γ is deductively closed. ■

[6] The names "existential" and "universal" come from the type of quantification in the standard valuation clauses for the non-classical connectives. Specifically, $\Diamond$ is an existential connective because $\Diamond\phi$ is true at a point w_i in a modal Kripke model iff there exists a point w_j in the model such that w_j is accessible from w_i and ϕ is true at w_j. Similarly, relevant implication $\Rightarrow$ is a universal connective because $\phi' \Rightarrow \phi''$ is true at a point w_i in a relevant Kripke model iff for all points w_j and w_k in the model, if w_i, w_j and w_k are in the compossibility relation and ϕ' is true at w_j, then ϕ'' is true at w_k.

We are now ready to introduce the canonical structure, whose points are built by partitioning Γ with respect to the labels.

Prop/Definition 75 Let $\langle \Sigma, D\rangle$ be an lds with congruence, and Γ a set of labelled formulae that is maximal with respect to γ. The *canonical structure* $\langle U, \mathcal{B}, \nu, \mu\rangle$ for Σ over Γ is defined as follows:

- $U = lb(\Gamma)$;
- $\mathcal{B} = \{\mathrm{Lab}(\phi) \mid \phi \in L(C, \Xi_{wf})\}$;
- ν is defined inductively as $\nu_0(\phi) = \mathrm{Lab}(\phi)$, and $\nu_k(c)(\mathrm{Lab}(\phi_1), \dots, \mathrm{Lab}(\phi_k)) = \mathrm{Lab}(c(\phi_1, \dots, \phi_k))$;
- $\mu_k(o) = \{(w, w_1, \dots, w_k) \mid w{:}o(w_1, \dots, w_k) \in \Gamma\}$.

Proof We check that ν is well-defined. Suppose $\mathrm{Lab}(\phi_i) = \mathrm{Lab}(\phi'_i)$ for $i = 1, \dots, k$. Then, by Lemma 74, for each $i = 1, \dots, k$, $\Gamma \cup \{w_1{:}\phi_i\} \vdash w_1{:}\phi'_i$ and $\Gamma \cup \{w_2{:}\phi'_i\} \vdash w_2{:}\phi_i$ for every $w_1, w_2 \in \Xi_{lb}$. Since $\langle \Sigma, D\rangle$ is an lds with congruence, for any $w_3, w_4 \in \Xi_{lb}$ we have that $\Gamma \cup \{w_3{:}c(\phi_1, \dots, \phi_k)\} \vdash w_3{:}c(\phi'_1, \dots, \phi'_k)$ and $\Gamma \cup \{w_4{:}c(\phi'_1, \dots, \phi'_k)\} \vdash w_4{:}c(\phi_1, \dots, \phi_k)$, and so, using again Lemma 74, we have $\mathrm{Lab}(c(\phi_1, \dots, \phi_k)) = \mathrm{Lab}(c(\phi'_1, \dots, \phi'_k))$. ■

The following lemma will help us to show that a canonical structure built this way is a structure for the labelled deduction system $\langle \Sigma, D\rangle$. The lemma consists of three statements. The first one says that the interpretation in a canonical structure of a schema formula under a given assignment is equal to the set of labels that decorate the schema formula (after a certain substitution dependent on the assignment) in the maximal set used to build the canonical structure.

The second statement says that checking the satisfaction of a labelled schema formula in a canonical structure under a given assignment is equivalent to checking that the maximal set originating the structure contains the labelled schema formula (again after a certain substitution dependent on the assignment).

The third statement relates satisfaction of a judgement in a canonical structure built from a maximal set to derivability from that set.

Lemma 76 Let $\langle \Sigma, D\rangle$ be an lds with congruence, Γ a set of labelled formulae that is maximal with respect to γ, and S the canonical structure for Σ over Γ. Then, for any assignment α and substitution σ such that σ^α is given by $\sigma^\alpha_{lb} = \alpha_{lb}$ and $\mathrm{Lab}(\sigma^\alpha_{wf}(\xi)) = \alpha_{wf}(\xi)$, we have:

1. $[\![\phi]\!]^{S,\alpha_{wf}} = \mathrm{Lab}(\phi\, \sigma^\alpha_{wf})$ for any schema formula $\phi \in L(C, \Xi_{wf})$.
2. $S, \alpha \Vdash \gamma'$ iff $\gamma'\, \sigma^\alpha \in \Gamma$ for any labelled schema formula $\gamma' \in L(\Sigma, \Xi)$.
3. For each judgement $\langle \Delta, \delta, \Phi\rangle \in J(\Sigma, \Xi)$ it holds: $S, \alpha \Vdash\!\sim \langle \Delta, \delta, \Phi\rangle$ iff $\Gamma \cup \Delta\, \sigma^{\alpha'} \vdash \delta\, \sigma^{\alpha'}$ for any $\alpha' \equiv_\Phi \alpha$.

Proof For 1, we proceed by induction on the structure of ϕ. (i) If $\phi \in C_0$, then $[\![\phi]\!]^{S,\alpha_{wf}} = \nu_0(\phi) = \mathrm{Lab}(\phi) = \mathrm{Lab}(\phi\, \sigma^\alpha_{wf})$. (ii) If $\phi \in \Xi_{wf}$, then $[\![\phi]\!]^{S,\alpha_{wf}} = \alpha_{wf}(\phi) =$

$\mathrm{Lab}(\phi\,\sigma^{\alpha}_{wf})$. (iii) If ϕ is $c(\phi_1,\dots,\phi_k)$, then $[\![\phi]\!]^{S,\alpha_{wf}} = \nu_k(c)([\![\phi_1]\!]^{S,\alpha_{wf}},\dots,[\![\phi_k]\!]^{S,\alpha_{wf}}) = \nu_k(c)(\mathrm{Lab}(\phi_1\,\sigma^{\alpha}_{wf}),\dots,\mathrm{Lab}(\phi_k\,\sigma^{\alpha}_{wf})) = \mathrm{Lab}(c(\phi_1\,\sigma^{\alpha}_{wf},\dots,\phi_k\,\sigma^{\alpha}_{wf})) = \mathrm{Lab}(\phi\,\sigma^{\alpha}_{wf})$.

For 2, we distinguish two cases, depending on whether or not φ in $\gamma' = w{:}\varphi$ is a relational schema formula. (i) If φ is $o(w_1,\dots,w_k)$, then we conclude since $S,\alpha \Vdash \gamma'$ iff $\alpha_{lb}(w) \in [\![o(w_1,\dots,w_k)]\!]^{S,\alpha_{lb}}$ iff $\alpha_{lb}(w) \in \mu_k(o)(\alpha_{lb}(w_1),\dots,\alpha_{lb}(w_k))$ iff $\alpha_{lb}(w) \in \mathrm{Lab}(o(\alpha_{lb}(w_1),\dots,\alpha_{lb}(w_k)))$ iff $\gamma'\,\sigma^{\alpha} \in \Gamma$. (ii) If γ' is $w{:}\varphi$ with $\varphi \in L(C,\Xi_{wf})$, then $S,\alpha \Vdash \gamma'$ iff $\alpha_{lb}(w) \in [\![\varphi]\!]^{S,\alpha_{wf}}$ iff $\alpha_{lb}(w) \in \mathrm{Lab}(\varphi\,\sigma^{\alpha}_{wf})$ iff $\alpha_{lb}(w){:}\varphi\,\sigma^{\alpha}_{wf} \in \Gamma$ iff $\gamma'\,\sigma^{\alpha} \in \Gamma$.

For 3, we have to show $S,\alpha \Vdash\!\!\sim \langle\Delta,\delta,\Phi\rangle$ iff $S,\alpha' \Vdash \delta$ whenever $S,\alpha' \Vdash \Delta$, for any $\alpha' \equiv_\Phi \alpha$. Using result 2. above this is equivalent to $\delta\,\sigma^{\alpha'} \in \Gamma$ whenever $\Delta\,\sigma^{\alpha'} \subseteq \Gamma$ for any $\alpha' \equiv_\Phi \alpha$. Now for any $\alpha' \equiv_\Phi \alpha$ consider two cases: (i) If $\Delta\,\sigma^{\alpha'} \subseteq \Gamma$, then $\delta\,\sigma^{\alpha'} \in \Gamma$ and since Γ is deductively closed, $\Gamma \cup \Delta\,\sigma^{\alpha'} \vdash \delta\,\sigma^{\alpha'}$. (ii) If $\Delta\,\sigma^{\alpha'} \not\subseteq \Gamma$, then $\Gamma \cup \Delta\,\sigma^{\alpha'} \vdash \delta\,\sigma^{\alpha'}$ since Γ is maximal. ■

We can now prove that canonical structures are indeed structures for lds's with congruence.

Lemma 77 Let $\langle\Sigma,D\rangle$ be an lds with congruence and Γ a set of labelled formulae that is maximal with respect to γ. The canonical structure S for Σ over Γ is a structure for $\langle\Sigma,D\rangle$.

Proof Let α be an assignment, and $r = \langle\{\langle\Delta_i,\delta_i,\Phi_i\rangle\}_{1\le i\le k},\langle\emptyset,\delta,\emptyset\rangle\rangle \in D$. Assume $S,\alpha \Vdash\!\!\sim \langle\Delta_i,\delta_i,\Phi_i\rangle$ for each $i=1,\dots,k$. So, for any $\alpha_i \equiv_{\Phi_i} \alpha$ we have, by Lemma 76, $\Gamma \cup \Delta_i\,\sigma^{\alpha_i} \vdash \delta_i\,\sigma^{\alpha_i}$, for each $i=1,\dots,k$. Therefore $\emptyset \vdash\!\!\sim \langle\Gamma \cup \Delta_i\,\sigma^{\alpha_i},\delta_i\,\sigma^{\alpha_i}\rangle$, $i = 1,\dots,k$. Using monotonicity, $\emptyset \vdash\!\!\sim \langle\Gamma \cup \Delta_i\,\sigma^{\alpha_i},\delta_i\,\sigma^{\alpha_i},\Phi_i\,\sigma^{\alpha_i}\rangle$, $i=1,\dots,k$, and so $\emptyset \vdash\!\!\sim \langle\Gamma,\delta\,\sigma^{\alpha}\rangle$ using rule r. So $\Gamma \vdash \delta\,\sigma^{\alpha}$, and $\delta\,\sigma^{\alpha} \in \Gamma$ since Γ is maximal. Therefore using Lemma 76 we have $S,\alpha \Vdash \delta$, and so $S,\alpha \Vdash\!\!\sim \langle\emptyset,\delta,\emptyset\rangle$. ■

This yields a completeness result for full lp's with congruence.

Theorem 78 Every full lp with congruence is complete.

Proof Let $\mathcal{L} = \langle\Sigma,M,A,D\rangle$ be a full lp with congruence, $\Gamma_0 \subseteq L(\Sigma,\Xi)$ and $\gamma \in L(\Sigma,\Xi)$. Assume $\gamma \notin \Gamma_0^{\vdash}$. Then, using our Henkin-style construction we can obtain a maximal extension Γ of Γ_0 with respect to γ. The canonical structure S for Σ over Γ is a structure for $\langle\Sigma,D\rangle$. Since $\mathcal{L}$ is full, there is a model $m \in M$ such that $A(m) = S$. So $m, id \not\Vdash \gamma$ since $\gamma \notin \Gamma$, and $m, id \Vdash \Gamma_0$ since $\Gamma_0 \subseteq \Gamma$. Thus $\gamma \notin \Gamma_0^{\vDash}$. ■

Having established the envisaged theorem on (the conditions for) completeness, we are now ready to deal with the target problem: under which conditions can we guarantee that completeness is preserved by fibring?

Assume that we are given two full labelled presentations, both with congruence, and, therefore, both complete. We have to address two questions: (i) is their (unconstrained/constrained) fibring an lp that is still full?, and (ii) is their (unconstrained/constrained) fibring an lp that is still with congruence? If the answer is affirmative in both cases, then we have established, by application of Theorem 78, that their (unconstrained/constrained) fibring is an lp that is still complete.

That is, if we are able to prove that both fullness and congruence are preserved by fibring, then we have established that so is completeness.

It is not difficult to adapt to our setting the proof for presentations based on Hilbert-style systems in [29, Theorem 4.6] of the fact that fullness is preserved by both forms of fibring. Indeed, the extra element μ in our interpretation structures raises no problem, and we have:

Lemma 79 Fullness of lp's is preserved by both forms of fibring. ■

Although congruence is not always preserved by fibring, we can again adapt the results in [29] to show that:

Lemma 80

(i) If a labelled presentation has equivalence, then it also has congruence.

(ii) Equivalence is preserved by both forms of fibring. ■

To explain this without going too much into details, it is worthwhile to recall that in [29] a Hilbert-style presentation of logic is said to have equivalence iff it has the implication and the equivalence connectives (possibly as abbreviations), and it satisfies the following metatheorems: the Metatheorem of Modus Ponens; the Metatheorem of Deduction; the Metatheorems of Biconditionality (relating implication with equivalence); and the Metatheorem of Substitution of Equivalents.

Rather than stating these metatheorems for Hilbert-style systems in full, which would require us to import too much notation and text from [29], we now illustrate the corresponding results for lds's and lp's.

Definition 81 An lds $\langle \Sigma, D \rangle$ is an *lds with implication* $\rightarrow$ iff (i) $\rightarrow \in C_2$ in Σ (possibly as an abbreviation), and (ii) D contains the following rules for implication elimination and introduction (possibly as derived rules):

$$\frac{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle \quad \langle \emptyset, w{:}\xi_1 \rangle}{\langle \emptyset, w{:}\xi_2 \rangle} \rightarrow \mathrm{E} \qquad \text{and} \qquad \frac{\langle \{w{:}\xi_1\}, w{:}\xi_2 \rangle}{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle} \rightarrow \mathrm{I}.$$

An lp is an *lp with implication* $\rightarrow$ iff its lds is an lds with implication $\rightarrow$. ■

In this case it is possible to state the Metatheorems of Modus Ponens and Deduction directly in the object logic, using the rules for $\rightarrow$. Namely, we immediately have

$$\{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle\} \mid\!\sim \langle \{w{:}\xi_1\}, w{:}\xi_2 \rangle$$

for the Metatheorem of Modus Ponens, and

$$\{\langle \{w{:}\xi_1\}, w{:}\xi_2 \rangle\} \mid\!\sim \langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle$$

for the Metatheorem of Deduction.

Primitive and derived rules are preserved by fibring since morphisms between lds's preserve judgement derivations, as we showed in Proposition 28. So, the fibring of lds's with implication is an lds with implication provided that the implication is shared. The same extends to lp's.

Definition 82 An lds $\langle \Sigma, D \rangle$ with implication $\rightarrow$ is an *lds with equivalence* $\leftrightarrow$ iff the following three conditions hold: (i) $\leftrightarrow \in C_2$ in Σ (possibly as an abbreviation), (ii) D contains the following rules for equivalence elimination and introduction (possibly as derived rules):

$$\frac{\langle \emptyset, w{:}\xi_1 \leftrightarrow \xi_2 \rangle}{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle} \leftrightarrow \mathrm{E1}\,, \qquad \frac{\langle \emptyset, w{:}\xi_1 \leftrightarrow \xi_2 \rangle}{\langle \emptyset, w{:}\xi_2 \rightarrow \xi_1 \rangle} \leftrightarrow \mathrm{E2}\,,$$

$$\frac{\langle \emptyset, w{:}\xi_1 \rightarrow \xi_2 \rangle \quad \langle \emptyset, w{:}\xi_2 \rightarrow \xi_1 \rangle}{\langle \emptyset, w{:}\xi_1 \leftrightarrow \xi_2 \rangle} \leftrightarrow \mathrm{I}\,,$$

and (iii) we can derive the following *Metatheorem of Substitution of Equivalents* (*MSE*)

$$\{\langle \emptyset, w{:}\xi_i \leftrightarrow \xi_i' \rangle \triangleright w\}_{i=1,\dots,k} \mid\!\sim \langle \emptyset, w'{:}c(\xi_1, \dots, \xi_k) \leftrightarrow c(\xi_1', \dots, \xi_k') \rangle$$

for every $k \in \mathbb{N}$ and $c \in C_k$.

An lp is an *lp with equivalence* $\leftrightarrow$ iff its lds is an lds with equivalence $\leftrightarrow$. ■

Like for implication, we can argue that the fibring of lds's with equivalence is an lds with equivalence provided that equivalence is shared, and that the same extends to lp's.

As an example, let us consider again the simple case of modal logics.

Example 83 The lds for mono-modal logic K of Example 10 extended with the connectives $\wedge$ and $\leftrightarrow$, defined as $\phi \wedge \phi' \equiv_{abrv} (\phi \rightarrow (\phi' \rightarrow \bot)) \rightarrow \bot$ and $\phi \leftrightarrow \phi' \equiv_{abrv} (\phi \rightarrow \phi') \wedge (\phi' \rightarrow \phi)$, is an lds with equivalence $\leftrightarrow$. This is because:

- $\rightarrow, \leftrightarrow \in C_2$;
- we have primitive rules for implication elimination and introduction;
- we can derive the rules $\leftrightarrow$ E1, $\leftrightarrow$ E2 and $\leftrightarrow$ I; and
- we can derive the MSE.

To illustrate the last point, we give the derivations for the cases in the MSE where c is $\rightarrow$ or $\Box$. For $\rightarrow$ we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle \Gamma, w'{:}\xi_2 \leftrightarrow \xi_2' \rangle}\,\mathrm{HYP}}{\langle \Gamma, w'{:}\xi_2 \rightarrow \xi_2' \rangle}\leftrightarrow\mathrm{E1} \quad \dfrac{\dfrac{}{\langle \Gamma, w'{:}\xi_1 \rightarrow \xi_2 \rangle}\,\mathrm{AX} \quad \dfrac{\dfrac{\dfrac{}{\langle \Gamma, w'{:}\xi_1 \leftrightarrow \xi_1' \rangle}\,\mathrm{HYP}}{\langle \Gamma, w'{:}\xi_1' \rightarrow \xi_1 \rangle}\leftrightarrow\mathrm{E2} \quad \dfrac{}{\langle \Gamma, w'{:}\xi_1' \rangle}\,\mathrm{AX}}{\langle \Gamma, w'{:}\xi_1 \rangle}\rightarrow\mathrm{E}}{\langle \Gamma, w'{:}\xi_2 \rangle}\rightarrow\mathrm{E}}{\langle \Gamma, w'{:}\xi_2' \rangle}\rightarrow\mathrm{E}}{\langle \{w'{:}\xi_1 \rightarrow \xi_2\}, w'{:}\xi_1' \rightarrow \xi_2' \rangle}\rightarrow\mathrm{I}}{\langle \emptyset, w'{:}(\xi_1 \rightarrow \xi_2) \rightarrow (\xi_1' \rightarrow \xi_2') \rangle}\rightarrow\mathrm{I} \quad \Pi(\leftarrow)}{\langle \emptyset, w'{:}(\xi_1 \rightarrow \xi_2) \leftrightarrow (\xi_1' \rightarrow \xi_2') \rangle}\leftrightarrow\mathrm{I}$$

where $\Gamma = \{w'{:}\xi_1 \rightarrow \xi_2, w'{:}\xi_1'\}$ and $\Pi(\leftarrow)$ is the derivation of $\langle \emptyset, w'{:}(\xi_1' \rightarrow \xi_2') \rightarrow (\xi_1 \rightarrow \xi_2) \rangle$, which has the same shape as that for the converse.

For $\Box$ we have:

$$
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  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\langle \Gamma, w'':\xi \leftrightarrow \xi' \rangle \rhd w''}\ \text{HYP}}{\langle \Gamma, w'':\xi \to \xi' \rangle \rhd w''}\ {\leftrightarrow}\text{E1}
        \qquad
        \dfrac{\dfrac{}{\langle \Gamma, w':\Box\xi \rangle \rhd w''}\ \text{AX} \qquad \dfrac{}{\langle \Gamma, w':\mathcal{R}\, w'' \rangle \rhd w''}\ \text{AX}}{\langle \Gamma, w'':\xi \rangle \rhd w''}\ \Box\text{E}
      }{\langle \Gamma, w'':\xi' \rangle \rhd w''}\ {\to}\text{E}
    }{\langle \{w':\Box\xi\}, w':\Box\xi' \rangle}\ \Box\text{I}
  }{\langle \emptyset, w':\Box\xi \to \Box\xi' \rangle}\ {\to}\text{I}
  \qquad \Pi(\leftarrow)
}{\langle \emptyset, w':\Box\xi \leftrightarrow \Box\xi' \rangle}\ {\leftrightarrow}\text{I}
$$

where $\Gamma = \{w':\Box\xi, w':\mathcal{R}\, w''\}$ and $\Pi(\leftarrow)$ is the derivation of $\langle \emptyset, w':\Box\xi' \to \Box\xi \rangle$, which has the same shape as that for the converse. ■

Proposition 84 The Metatheorem of Congruence holds in an lds with equivalence.

Proof Let $\langle \Sigma, D \rangle$ be an lds with equivalence $\leftrightarrow$. Assume $\Gamma \cup \{w_{i_1}:\phi_i\} \vdash w_{i_1}:\phi'_i$ and $\Gamma \cup \{w_{i_2}:\phi'_i\} \vdash w_{i_2}:\phi_i$ for each $i = 1, \ldots, k$ and for any $w_{i_1}, w_{i_2} \in \Xi_{lb}$. Then using Lemma 25 (monotonicity of $\mid\!\sim$) there are derivation trees Π_i for $\emptyset \mid\!\sim \langle \Gamma \cup \{w':c(\phi_1, \ldots, \phi_k), w:\phi_i\}, w:\phi'_i \rangle \rhd w$ and Π'_i for $\emptyset \mid\!\sim \langle \Gamma \cup \{w':c(\phi_1, \ldots, \phi_k), w:\phi'_i\}, w:\phi_i \rangle \rhd w$ for each $i = 1, \ldots, k$. Thus, for each $k \in \mathbb{N}$ and $c \in C_k$ we can construct the following derivation tree for $\emptyset \mid\!\sim \langle \Gamma', w':c(\phi'_1, \ldots, \phi'_k) \rangle$ where $\Gamma' = \Gamma \cup \{w':c(\phi_1, \ldots, \phi_k)\}$:

$$
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  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\begin{matrix}\Pi_i \\ \langle \Gamma' \cup \{w:\phi_i\}, w:\phi'_i \rangle \rhd w\end{matrix}}{\langle \Gamma', w:\phi_i \to \phi'_i \rangle \rhd w}\ {\to}\text{I}
        \qquad
        \dfrac{\begin{matrix}\Pi'_i \\ \langle \Gamma' \cup \{w:\phi'_i\}, w:\phi_i \rangle \rhd w\end{matrix}}{\langle \Gamma', w:\phi'_i \to \phi_i \rangle \rhd w}\ {\to}\text{I}
      }{\langle \Gamma', w:\phi_i \leftrightarrow \phi'_i \rangle \rhd w}\ {\leftrightarrow}\text{I}
    }{\langle \Gamma', w':c(\phi_1, \ldots, \phi_k) \leftrightarrow c(\phi'_1, \ldots, \phi'_k) \rangle}\ \text{MSE}
  }{\langle \Gamma', w':c(\phi_1, \ldots, \phi_k) \to c(\phi'_1, \ldots, \phi'_k) \rangle}\ {\leftrightarrow}\text{E1}
  \qquad
  \dfrac{}{\langle \Gamma', w':c(\phi_1, \ldots, \phi_k) \rangle}\ \text{AX}
}{\langle \Gamma', w':c(\phi'_1, \ldots, \phi'_k) \rangle}\ {\to}\text{E}
$$

■

This allows us to summarize our results as follows.

Theorem 85 Completeness is preserved by the (unconstrained/constrained) fibring of two full lp's sharing equivalence.

Proof Fullness is preserved by both forms of fibring. The fibring also has equivalence, and hence it has congruence. Since every full lp with congruence is complete, the fibring is complete. ■

Note that, as already pointed out in [29], the class of logics with equivalence is very wide, subsuming all extensions of *basic logic* [24] (e.g. classical, intuitionistic, modal, linear and quantum logic). Thus, the sufficient condition established above for the preservation of completeness by fibring does cover most useful cases.

5 Concluding remarks

As part of ongoing research on the theory of fibring and on the theory (and practice) of labelled deduction systems, we have given here an account of how labelled deduction systems for logics with a propositional basis behave under fibring. We have first defined what is a logic presented with a labelled deduction system, and then given a categorial definition of (unconstrained and constrained by symbol sharing) fibring of labelled deduction systems. We have then adapted general semantics to the case of labelled deduction systems, defined fibring at the semantic level (i.e. fibring of interpretation systems), and established the conditions under which our labelled presentations are sound and complete. Finally, we have established requirements on the labelled presentations so that completeness is preserved by fibring.

While our work here builds on our previous experiences with labelled deduction systems and with fibring of Hilbert-style systems, and we were able to import some of the results we obtained previously, it is worthwhile to observe that we tried to give a general approach. We have given a novel (algebraic) presentation of labelled deduction systems which provides us with a suitable basis for defining their fibring, and which subsumes our previous work as a simple special case.

The problem of combining logics has been attracting much attention, and our work was inspired by the pioneering research on fibring by Gabbay and colleagues [5, 9, 12, 15, 17, 18, 20]. For example, in [5], Gabbay and Beckert first identify the conditions that allow tableaux systems to be well-suited for fibring, and then provide a sound and complete system for the fibring of two logics passing from tableaux of one component to tableaux of the other, and vice versa. Along the same line of work, in [20] Gabbay and Governatori use fibring as a way to combine modal logics presented using the labelled tableaux system KEM; they adapt KEM in order to obtain a modular and flexible tableaux-like proof method for the multi-modal logics arising from the fibring of modal logics. We believe that our notions and results can be extended easily to both tableaux and sequent systems, but investigating this in more detail will be a subject for future work.

It is also important to stress that in this paper we provide a characterization of the lds's/lp's resulting from the fibring in terms of explicit models and explicit deduction systems. We also analyze preservation by fibring of soundness and completeness, and of some metatheoretical properties such as congruence. As we remarked above, preservation of other metatheoretical properties such as the finite model property and decidability will also be a subject for future work

Besides the investigation of fibring of other Gentzen-style deduction systems and the preservation of other metatheoretical properties, throughout the paper we have already indicated also other directions for future research. Of particular interest are the extensions of the propositional language of the labelled deduction systems (removing the restrictions on relational formulae and allowing the use of hybrid formulae), and the extensions to the first-order case with terms and binding operators. But the most interesting direction, in our opinion, is the one aiming at developing the theory of fibring of logics endowed with heterogeneous deduction systems and/or semantics, where, for example, we are allowed to fibre a labelled deduction system with a Hilbert-style system, or a Kripke-style semantics with a Tarskian one.

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