Categorical Foundations for Randomly Timed Automata

P. Mateus [a,1] M. Morais [b,2] C. Nunes [b,2] A. Pacheco [b,2]
A. Sernadas [a,1] C. Sernadas [a,1]

[a] *Center for Logic and Computation / CMA, Dep. Matemática, IST, Portugal*

[b] *Stochastic Processes Group, CMA, Dep. Matemática, IST, Portugal*

Abstract

The general theory of randomly timed automata is developed: starting with the practical motivation and presentation of the envisaged notion, the categorical theory of minimization, aggregation, encapsulation, interconnection and realization of such automata is worked out. All these constructions are presented universally: minimization and realization as adjunctions, aggregation as product, interconnection as cartesian lifting, and encapsulation as co-cartesian lifting. Stochastic timed automata are shown to be a particular case of randomly timed automata. The notion of stochastic timed automaton is shown to be too restrictive to establish a self contained theory of combination and realization.

Contents

[1] Partially supported by *Fundação para a Ciência e a Tecnologia*, FEDER Project POCTI/2001/MAT/37239 FibLog, PRAXIS XXI Projects PRAXIS/P/MAT/10002/1998 ProbLog and 2/2.1/TIT/1658/95 LogComp, as well as by the ESPRIT IV Working Groups 22704 ASPIRE and 23531 FIREworks.

[2] Partially supported by *Fundação para a Ciência e a Tecnologia* and the PRAXIS XXI Project PRAXIS/P/MAT/10002/1998 ProbLog.

1 Introduction

Probabilistic systems and stochastic models of computing have been attracting much attention in recent years [23,36,33,7,14,3,4,27]. In each case, an abstract notion of stochastic machine is established by endowing a classical notion of automaton or transition system with a specific probabilistic mechanism. By starting with the classical notion of timed automaton [20,13] and regarding the action times as random variables, one reaches the notion of *randomly timed automata*, as considered for example in [9]. By endowing Petri nets with random transition times, one ends up with the related notion of stochastic Petri nets, as considered for example in [17,24,8].

The main goal of this paper is the development of a suitably general theory of randomly timed automata, covering the following aspects: minimization, aggregation and interconnection, encapsulation, realization (obtaining the stochastic point process of runs). The required notion of randomly timed automaton should be as general as possible in order to be able to support different execution policies. Different policies are studied in [24,8] for the case of stochastic Petri nets, but their usefulness extends to randomly timed automata as well.

The notion of randomly timed automaton is well justified as an extension of the notion of stochastic timed automaton [10], where the categorical theory of unfolding [2] and combination can be established smoothly. As we shall see, the concept of stochastic timed automaton is too restrictive to set up such a powerful theory, since in order to cover all desirable cases of combination we need to establish dependencies between the random times of actions, which is not possible in the stochastic timed automata setting.

Following the style of [19,15,5,1,18,6,2,35,32] proposed for classical automata, we adopt a categorical approach to the development of the theory of randomly timed automata. However, probabilistic structures raise category theoretic problems [26,31], as the desirable notion of morphism does not behave well with respect to composition. The analysis of this problem leads to the notion of precategory, a structure weaker than a category. Fortunately, we are able to avoid working with precategories as advocated in these two papers, by working with families of random sources. In this way we manage to stay within standard category theory.

Besides assuming that the reader is conversant with elementary category theory in order to follow the categorical development of the theory of randomly timed automata, we also assume that the reader is at ease with the basic concepts of abstract probability theory and stochastic processes. Point processes are used, but no deep result about them is needed. The text book [12] is an excellent source about point processes.

The paper is organized as follows. Section 2 starts with some motivating examples before the central notion of randomly timed automaton is introduced, followed immediately by the appropriate notion of morphism. Section 3 is dedicated to the minimization problem: within a fiber, minimal randomly timed automata are shown to constitute a co-reflective subcategory. Section 4 addresses the issue of combining, interconnecting and hiding actions in randomly timed automata: all these constructions are shown to be universal (products, cartesian and co-cartesian liftings). Section 5 deals with realization: the unfolding and folding functors are shown to establish an adjunction between the category of behaviours (point processes) and a suitable subcategory of randomly timed automata.

2 Randomly timed automata

We start by considering a few examples of what we would like to consider as randomly timed automata. Such a machine should basically be a timed automaton where the action times are random variables. At a given configuration, there is a race between the different actions: the first action(s) to occur trigger the corresponding random transition in the system (typically this transition is non-deterministic). Furthermore, in order to be able to model in-

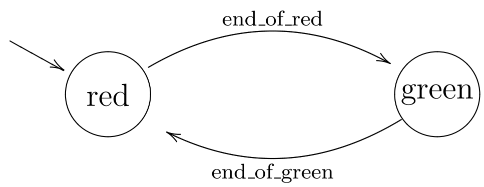

Fig. 1. Ideal semaphore

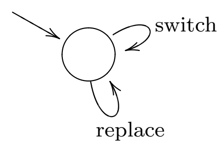

Fig. 2. Lightbulb

terconnection, it is essential to allow the possibility of a transition (triggered by the environment) before the occurrence of the first action(s).

Example 1 *Ideal semaphore.* Consider a simple semaphore with two states ('red' and 'green') and two actions ('end_of_red' and 'end_of_green'), as depicted in Figure 1. In this case, a configuration should be a pair composed of a state and a nonnegative real number. The latter represents the sojourn time in the former. Once in a state, the time to the occurrence of each action is a positive random variable with some distribution (for instance exponential, but we would like to be able to cope with other distributions, with memory).

In the next example, the impact of the chosen execution policies is already very clear: what happens to the other random variables when one of the actions occurs (wins the race)? They may be reset or not, depending on the application at hand.

Example 2 *Lightbulb.* Consider a simple lightbulb with a single state and two actions ('switch' and 'replace'), as shown in Figure 2. The former corresponds to an abstraction of switch on/off. The latter corresponds to replacing the lightbulb. Assume that we have one state and the aging of the lightbulb is not reset by a switching and vice-versa. Then a configuration should give, for each action, the interval of time since the action has occurred. Again, the time to each action is a positive random variable with some appropriate distribution.

Finally, consider the following example where the configurations are fully representative of the general notion.

Example 3 *Semaphore.* The semaphore represented in Figure 3 is similar to

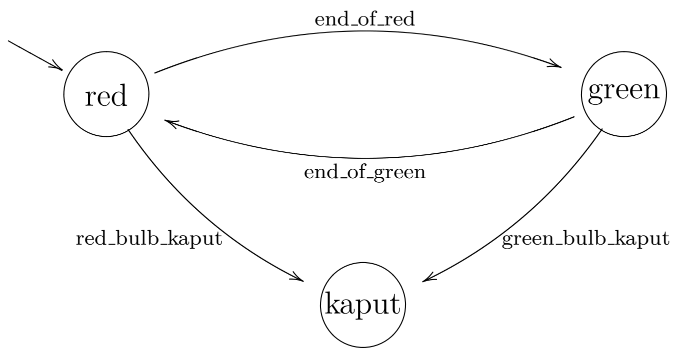

Fig. 3. Semaphore

the simpler one described in Example 3, but it contains a third state (‘kaput’) corresponding to being nonfunctional. The semaphore moves to that state by blowing the red lightbulb (‘red_bulb_kaput’) or the green lightbulb (‘green_bulb_kaput’). If each lightbulb ages only when it is on, the random time for red (green) bulb kaput depends on the cumulative time the red (green) bulb has been on. So, we must obtain from a configuration the cumulative time each bulb has been on.

Before proceeding we will review some basic notions of probability theory that we use extensively in the sequence.

2.1 Basic background on probability theory

Herein we introduce some basic concepts of probability theory. For further details refer to [30].

A function $F : \mathbb{R} \cup \{+\infty\} \to [0, 1]$ is a cumulative distribution function if: (*i*) It is a non-decreasing function; (*ii*) It is right-continuous and has left-hand limits; and (*iii*) $\lim_{x\to-\infty} F(x) = 0$ and $F(+\infty) = 1$.

We denote by $\mathcal{L}_{(0,1]} = \langle (0, 1], \mathbb{B}_{(0,1]}, \mu\rangle$ the probability space where $\mathbb{B}_{(0,1]}$ is the Borel σ-algebra over $(0, 1]$ and μ is the Lebesgue measure. Note that, if U is a random variable over $\mathcal{P} = \langle \Omega, \mathcal{F}, P\rangle$ with uniform distribution in (0,1] and F is a cumulative distribution function, then the random variable $F^{-1} \circ U$ (over $\mathcal{P}$) with $F^{-1}(u) = \inf\{x \in \mathbb{R} \cup \{+\infty\} : F(x) \geq u\}$

$$\Omega \xrightarrow{U} (0,1] \underset{F}{\overset{F^{-1}}{\rightleftarrows}} \mathbb{R} \cup \{+\infty\}$$

has cumulative distribution function F. Furthermore, observe that if $\mathcal{P} = \mathcal{L}_{(0,1]}$ and U is the identity then U has uniform distribution in $(0, 1]$, and so the random variable $F^{-1} : (0, 1] \to \mathbb{R} \cup \{+\infty\}$ has cumulative distribution F.

Given a positive random variable X over $\mathcal{L}_{(0,1]}$ with cumulative distribution function F and a positive number x, we define the random variable $X|_x$, called the *residual lifetime of X at age x random variable* over $\mathcal{L}_{(0,1]}$ by

$$X|_x(\omega) = F^{-1}((1 - F(x))\,\omega + F(x)) - x.$$

Thus, the cumulative distribution function of the residual lifetime of X at age x is given by

$$P(X|_x \leq y) = P(X \leq x + y | X > x) = \frac{F(x + y) - F(x)}{1 - F(x)}.$$

Given a probability space $\langle \Omega, \mathcal{F}, P \rangle$ and a set Γ, the family $\{X(t) : t \in \Gamma\}$ is a stochastic process (with parameter set Γ) if $X(t)$ is a random variable over $\langle \Omega, \mathcal{F}, P \rangle$, for all $t \in \Gamma$. A (stochastic) point process is an (almost surely) non-decreasing sequence $(T_n)_{n \geq 1}$ of nonnegative random variables T_n. If T_n is finite, then T_n may be interpreted as the n-th *point* or *atom* of the process.

2.2 Objects

Building up on the motivation gained from the examples above we are ready to propose the envisaged general notion of randomly timed automaton.

Definition 4 A *randomly timed automaton (rta)* is a tuple $\langle \Sigma, \mathcal{P}, A, \Xi, T, \delta \rangle$ where:

- Σ is a countable set (of *random sources*);
- $\mathcal{P} = \{\mathcal{P}_\sigma\}_{\sigma \in \Sigma}$ where each $\mathcal{P}_\sigma = \langle \Omega_\sigma, \mathcal{F}_\sigma, P_\sigma \rangle$ is a probability space;
- A is a finite set (of *actions*);
- Ξ is a pointed set (of *configurations*) with distinguished element ξ_0 (the *initial configuration*);
- $T = \{T^\xi\}_{\xi \in \Xi}$, where each T^ξ is an A-indexed family of extended positive random variables over the product probability space $\mathcal{P}^\bullet = \prod_{\sigma \in \Sigma} \mathcal{P}_\sigma = \langle \Omega^\bullet, \mathcal{F}^\bullet, P^\bullet \rangle$ (that is, a random vector taking values in $(\mathbb{R}_+^\infty)^A$ with $\mathbb{R}_+^\infty = (0, +\infty]$);
- $\delta = \{\delta^\xi\}_{\xi \in \Xi}$, where each δ^ξ is a stochastic process over $\mathcal{P}^\bullet$ with

  $$\delta^\xi(\omega) : (0, \mu^\xi(\omega)] \cap \mathbb{R} \to \Xi$$

  where $\omega \in \Omega^\bullet$ and $\mu^\xi(\omega) = \inf\{T_a^\xi(\omega) : a \in A\}$.

When considering stochastic processes, it usually the case that the process depends on several independent random sources. This is equivalent to saying that the process depends on several probability spaces. For instance, in queuing systems, the queue size depends on the customer arrival and the service times sources. These two sources are assumed to be independent. Thus, the queue size process has to be defined over the product of these two random sources. If we want to model the queue size process using an rta, we have to consider two probability spaces, one associated to customer arrivals and another one associated to service times. This justifies the inclusion in the definition of a family of probability spaces, assumed to be independent so that we can work in their product.

The random family T^ξ gives us the *random times* for each action from configuration ξ. These random families are defined over the product probability space $\mathcal{P}^\bullet$. The idea of having multiple random sources comes from the need of working with the combination of rta's that may have been established separately over different probability spaces. The *random transition function* δ^ξ gives us, from configuration ξ and for each *admissible cut instant* t (in $(0, \mu^\xi(\omega)]$), the random next configuration. Note that the random transition function describes the transition before the occurrence of the first action(s). We say that an admissible cut instant t is *proper* whenever $t < \mu^\xi(\omega)$. Starting from ξ, there is a race mechanism between the actions. If no cut is made before, the rta jumps to the next configuration when the winning action(s) happen (at *random winning time* μ^ξ).

In the sequel, we shall use the following auxiliary notation: (i) The *random set of winning actions* from configuration ξ is a random quantity over $\mathcal{P}^\bullet$ with

$$\alpha^\xi : \Omega^\bullet \to \wp A$$

where the random set of winning actions from configuration ξ is

$$\alpha^\xi(\omega) = \{a \in A : T^\xi_a(\omega) = \mu^\xi(\omega)\}.$$

(ii) The *random extended transition function* at a sequence of cut instants $\vec{t}$ from configuration ξ is inductively defined as follows:

$$\begin{aligned} &\delta^{*\xi}(\epsilon)(\epsilon) = \xi; \\ &\delta^{*\xi}(\vec{\omega}\omega)(\vec{t}t) = \delta^{\delta^{*\xi}(\vec{\omega})(\vec{t})}(\omega)(t). \end{aligned}$$

Note that the sequences $\vec{\omega}$ and $\vec{t}$ must be of the same length. Furthermore, at each step, the cut instant must be within the admissible interval. For instance, for the last step, t must be in $(0, \mu^{\delta^{*\xi}(\vec{\omega})(\vec{t})}(\omega)]$. That is, $\vec{t}$ must belong to the set $\mathrm{adm}^\xi(\vec{\omega})$ of admissible cut vectors from ξ for $\vec{\omega}$:

$$\begin{aligned} &\mathrm{adm}^\xi(\epsilon) = \{\epsilon\}; \\ &\mathrm{adm}^\xi(\omega\vec{\omega}) = \{t\vec{t} : t \in (0, \mu^\xi(\omega)] \ \ \& \ \ \vec{t} \in \mathrm{adm}^{\delta^\xi(\omega)(t)}(\vec{\omega})\}. \end{aligned}$$

Example 5 We present the ideal semaphore sketched in Example 1 as an rta. Let F_{eor} and F_{eog} be the cumulative distribution functions associated to the sojourn times in the green and red states in each visit, respectively. The semaphore is described as the rta $\langle \Sigma, \mathcal{P}, \Xi, \xi_0, T, \delta \rangle$ where:

- $\Sigma = \{*\}$; for this example we require only one random source;
- $\mathcal{P}_* = \mathcal{L}_{(0,1]} = \langle (0,1], \mathbb{B}_{(0,1]}, \mu \rangle$; as shown in Section 2.1, in this probability space we are able to generate any cumulative distribution function of a random time for an action;
- $A = \{\text{eor}, \text{eog}\}$ where *eor* denotes the *end_of_red* action and *eog* denotes the *end_of_green* action;
- $\Xi = S \times \mathbb{R}_0^+$ where $S = \{\text{red}, \text{green}\}$; the configuration $\langle s, t \rangle$ indicates that the light s is on and has been on for t units of time;
- $\xi_0 = \langle \text{red}, 0 \rangle$; we suppose that the semaphore starts with the red light on;
- The transitions and random times for actions from configurations where the light is *red* are given by:
  · $T_{\text{eor}}^{\langle \text{red},0 \rangle}(\omega) = F_{\text{eor}}^{-1}(\omega)$; as explained in the appendix $T_{\text{eor}}^{\langle \text{red},0 \rangle}$ has cumulative distribution function F_{eor};
  · $T_{\text{eor}}^{\langle \text{red},z \rangle} = T_{\text{eor}}^{\langle \text{red},0 \rangle}|_z$ for $z > 0$, where $T_{\text{eor}}^{\langle \text{red},0 \rangle}|_z$; thus, as explained in Section 2.1, the cumulative distribution function of $T_{\text{eor}}^{\langle \text{red},z \rangle}$ is the residual time distribution of $T^{\langle \text{red},0 \rangle}$ at age z;
  · $T_{\text{eog}}^{\langle \text{red},z \rangle}(\omega) = +\infty$; so *eog* does not occur when the light is *red*;
  · $\delta^{\langle \text{red},z \rangle}(\omega)(t) = \begin{cases} \langle \text{green}, 0 \rangle & \text{provided that } t = T_{\text{eor}}^{\langle \text{red},z \rangle}(\omega) \\ \langle \text{red}, z+t \rangle & \text{otherwise} \end{cases}$;

  thus *eor* action triggers the transition of the light from *red* to *green*; if *eor* action does not occur, the elapsed time on *red* is increased continuously as time runs;
- The transitions and random times for actions from configurations where the the semaphore is in state *green* are similar to the *red* case.

So, the random winning time at configuration $\langle \text{red}, z \rangle$, $\mu^{\langle \text{red},z \rangle}$, is equal to $T_{\text{eor}}^{\langle \text{red},z \rangle}$ and the random winning time at configuration $\langle \text{green}, z \rangle$, $\mu^{\langle \text{green},z \rangle}$, is equal to $T_{\text{eog}}^{\langle \text{green},z \rangle}$. Furthermore, the random set of winning actions at configuration $\langle \text{red}, z \rangle$, $\alpha^{\langle \text{red},z \rangle}$, is deterministic and equal to $\{\text{eor}\}$ and the random set of winning actions at configuration $\langle \text{green}, z \rangle$, $\alpha^{\langle \text{green},z \rangle}$, is also deterministic and equal to $\{\text{eog}\}$.

Example 6 We present the semaphore sketched in Example 3 as an rta. Let F_{eor} and F_{eog} be as in Example 5, and let F_{rbk} and F_{gbk} be the cumulative distribution functions of the lifetimes of the green and red bulbs, respectively. The semaphore is described by the rta $\langle \Sigma, \mathcal{P}, \Xi, \xi_0, T, \delta \rangle$ where:

- $\Sigma = \{\text{s}, \text{rk}, \text{gk}\}$; for this example we require one random source for the switching between lights (s), one for the *red_bulb_kaput* (rk) and one for the *green_bulb_kaput* (gk). This imposes that these three random mechanisms are independent;

- $\mathcal{P}_{\mathrm{s}} = \mathcal{P}_{\mathrm{rk}} = \mathcal{P}_{\mathrm{gk}} = \mathcal{L}_{(0,1]} = \langle (0,1], \mathbb{B}_{(0,1]}, \mu \rangle$;
- $A = \{\mathrm{eor}, \mathrm{eog}, \mathrm{rbk}, \mathrm{gbk}\}$ where *eor* denotes the *end_of_red* action, *eog* denotes the *end_of_green*, *rbk* denotes the *red_bulb_kaput* and *gbk* denotes the *green_bulb_kaput*;
- $\Xi = S \times \mathbb{R}_0^{+3}$ where $S = \{\mathrm{red}, \mathrm{green}, \mathrm{kaput}\}$ the configuration $\langle s, t, t_r, t_g \rangle$ indicates that the semaphore has remained in state s in the last t units of time and that the cumulative amount of time the red (green) light has been on is t_r (t_g);
- $\xi_0 = \langle \mathrm{red}, 0, 0, 0 \rangle$; we suppose that the semaphore starts with the red light on and with new bulbs;
- The transitions and random times for actions from configurations where the semaphore is in the *red* state:
  - $T_{\mathrm{eor}}^{\langle \mathrm{red},0,y,z \rangle}(\omega) = F_{\mathrm{eor}}^{-1}(\omega)$;
  - $T_{\mathrm{eor}}^{\langle \mathrm{red},x,y,z \rangle} = T_{\mathrm{eor}}^{\langle \mathrm{red},0,y,z \rangle}|_x$ for $x > 0$;
  - $T_{\mathrm{rbk}}^{\langle \mathrm{red},x,0,z \rangle}(\omega) = F_{\mathrm{rbk}}^{-1}(\omega)$;
  - $T_{\mathrm{rbk}}^{\langle \mathrm{red},x,y,z \rangle} = T_{\mathrm{rbk}}^{\langle \mathrm{red},x,0,z \rangle}|_y$ for $y > 0$;
  - $T_{\mathrm{eog}}^{\langle \mathrm{red},x,y,z \rangle}(\omega) = T_{\mathrm{gbk}}^{\langle \mathrm{red},x,y,z \rangle}(\omega) = +\infty$;
  - $\delta^{\langle \mathrm{red},x,y,z \rangle}(\omega)(t) = \begin{cases} \langle \mathrm{kaput}, 0, y+t, z \rangle & \mathrm{pt}\ t = T_{\mathrm{rbk}}^{\langle \mathrm{red},x,y,z \rangle}(\omega) \\ \langle \mathrm{green}, 0, y+t, z \rangle & \mathrm{pt}\ t = T_{\mathrm{eor}}^{\langle \mathrm{red},x,y,z \rangle}(\omega) < T_{\mathrm{rbk}}^{\langle \mathrm{red},x,y,z \rangle}(\omega) \\ \langle \mathrm{red}, x+t, y+t, z \rangle & \mathrm{otherwise} \end{cases}$;

    thus the *rbk* action triggers the transition of the light from *red* to *kaput*; the *eor* action triggers the transition of the light from *red* to *green*; and while the *eor* and *rbk* actions do not occur, the elapsed time on *red* is increased continuously as time runs;
- The transitions and random times for actions from configurations where the the semaphore is in state *green* are similar to the *red* case.
- The transitions and random times for actions from configurations where the semaphore is in the *kaput* state:
  - $T_{\mathrm{eor}}^{\langle \mathrm{kaput},x,y,z \rangle}(\omega) = T_{\mathrm{eog}}^{\langle \mathrm{kaput},x,y,z \rangle}(\omega) = T_{\mathrm{rbk}}^{\langle \mathrm{kaput},x,y,z \rangle}(\omega) = T_{\mathrm{gbk}}^{\langle \mathrm{kaput},x,y,z \rangle}(\omega) = +\infty$;
  - $\delta^{\langle \mathrm{kaput},x,y,z \rangle}(\omega)(t) = \langle \mathrm{kaput}, x+t, y, z \rangle$;

2.3 Stochastic timed automata

One of the most important classes of discrete event systems is that of *stochastic timed automata*. We consider an outcome presentation of these automata corresponding to the notion presented in [10] §6.4.

The main differences between an rta and an sta are that in the latter, all random times for the actions are assumed to be independent, and their memory must be encoded into a set of countable states.

Definition 7 A *stochastic timed automaton (sta)* is a tuple $\langle \Sigma, \mathcal{P}, A, S, \Gamma, X, \Delta \rangle$ where:

- Σ is a countable set (of *random sources*);
- $\mathcal{P} = \{\mathcal{P}_\sigma\}_{\sigma\in\Sigma}$ where each $\mathcal{P}_\sigma = \langle \Omega_\sigma, \mathcal{F}_\sigma, P_\sigma \rangle$ is a probability space;
- A is a finite set (of *actions*);
- S is a countable pointed set (of *states*) with distinguished element s_0
- $\Gamma = \{\Gamma^s\}_{s\in S}$ where each $\Gamma^s \subseteq A$;
- $X = \{X_a\}_{a\in A}$ where each X_a is a positive random variable over $\mathcal{L}_{(0,1]} = \langle (0,1], \mathbb{B}_{(0,1]}, \mu \rangle$;
- $\Delta = \{\Delta^s\}_{s\in S}$ where each Δ^s is a $(\wp(\Gamma^s) \setminus \{\emptyset\})$-indexed family of random quantities taking values in S defined over $\mathcal{P}^\bullet = \prod_{\sigma\in\Sigma} \mathcal{P}_\sigma = \langle \Omega^\bullet, \mathcal{F}^\bullet, P^\bullet \rangle$ (that is, measurable maps $\Delta^s_E : \Omega^\bullet \to S$ for each $s \in S$ and $E \in (\wp(\Gamma^s) \setminus \{\emptyset\})$.

The set Γ^s gives us the set of enabled actions in state s. The random variable X_a gives the time until action a occurs. Observe that the random variables X_a's are independent of each other.

Given a state s and the age of each action $\{x_a\}_{a\in A}$ with $x_a \in \mathbb{R}^+_0$, there is a race between all enabled actions at state s and the random winning time is given by

$$m^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A) = \inf_{a\in\Gamma^s} \{X_a|_{x_a}(\omega_a)\}$$

where $\omega_A = \{\omega_a\}_{a\in A} \in (0,1]^A$ and $X_a|_{x_a}(\omega_a)$ is the residual time random variable of action a at age x_a. Given the random set of winning actions

$$W^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A) = \{a \in \Gamma^s : X_a|_{x_a}(\omega_a) = m^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A)\}$$

we obtain the next state through the random transition $\Delta^s_{W^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A)}$. After this transition the age of the actions becomes $y^{\langle s,\{x_a\}_{a\in A}\rangle} = \{y_b^{\langle s,\{x_a\}_{a\in A}\rangle}\}_{b\in A}$ where

$$y_b^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A) = \begin{cases} x_b + m^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A) & \text{pt } b \notin W^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A) \,\&\, b \in \Gamma^s \\ 0 & \text{otherwise} \end{cases}.$$

Obviously, we start at state s_0 with the age of each action being zero.

Remark 8 Our notion of sta differs from the notion presented in [10] in the following aspects:

- We consider a finite set of events, and not a countable set. By not assuming the finiteness constraint the following problems may happen:
  - · the random set of winning actions may be empty since $\min_{a\in\Gamma^s}\{X_a|_{x_a}\}$ may not exist.

  · the random winning time $\inf_{a\in\Gamma^s}\{X_a|_{x_a}\}$ may take the value zero. Note, for instance, that if $\{X_1, ...X_n, ..\}$ is a sequence of independent and identically distributed positive random variables and $P(X_1 < \epsilon) > 0$ for all $\epsilon > 0$ then $P(\inf_{i\in\mathbb{N}} X_i = 0) = 1$.

  These problems can be fixed by imposing that Γ^s is finite for all $s \in S$.
- We considered the more general case where Δ depends on the set of winning events instead of just the winning event with the highest priority.
- To be consistent with the approach followed for rta's we presented the less general case where the initial state s_0 is constant and not a random quantity over S. All the theory presented in this paper may be straightforwardly extended for the latter case.

The ideal semaphore sketched in Example 1 can also be presented as an sta.

Example 9 Let F_{eor} and F_{eog} be the cumulative distribution functions associated to the sojourn times in the green and red states in each visit, respectively. The semaphore is described as the sta $\langle\{*\}, \mathcal{P}, A, S, \Gamma, X, \Delta\rangle$ where:

- $\mathcal{P}_* = \langle\{*\}, \{\emptyset, \{*\}\}, P_*\rangle$ is the trivial probability space;
- $A = \{\text{eor}, \text{eog}\}$ where *eor* denotes the *end_of_red* action and *eog* denotes the *end_of_green* action;
- $S = \{\text{red}, \text{green}\}$; the state indicates which light is on;
- $s_0 = \text{red}$; we suppose that the semaphore starts with the red light on;
- $\Gamma^{\text{red}} = \{\text{eor}\}$ and $\Gamma^{\text{green}} = \{\text{eog}\}$;
- $X_{\text{eor}}(\omega) = F^{-1}_{\text{eor}}(\omega)$ and $X_{\text{eog}}(\omega) = F^{-1}_{\text{eog}}(\omega)$;
- $\Delta^{\text{red}}_{\{\text{eor}\}}(*) = \text{green}$ and $\Delta^{\text{green}}_{\{\text{eog}\}}(*) = \text{red}$.

So, the random winning time at state *red* and age times $\{x_{\text{eor}}, x_{\text{eog}}\}$, denoted by $m^{\langle\text{red},\{x_{\text{eor}},x_{\text{eog}}\}\rangle}$, is equal to $X_{\text{eor}}|_{x_{\text{eor}}}$ and so, the random set of winning actions, $W^{\langle\text{red},\{x_{\text{eor}},x_{\text{eog}}\}\rangle}$, is deterministic and equal to $\{\text{eor}\}$. The next state $\Delta^{\text{red}}_{\{\text{eor}\}}(*)$ is *green*, after this transition the ages of both actions are set to zero. The case when the state is *green* is similar.

Note that the semaphore sketched in Example 3 can not be presented as an sta, unless the lifetimes of the bulbs have exponential distributions, since it is impossible to record the cumulative amount of time a bulb has been on using the sta framework (recall that the state space of an sta is countable).

We now show how to extract an rta from an sta.

Proposition 10 Given an sta $n = \langle\Sigma, \mathcal{P}, A, S, \Gamma, X, \Delta\rangle$ we can extract an rta

$$E(n) = \langle\Sigma', \mathcal{P}', A, \Xi, T, \delta\rangle$$

where:

- $\Sigma' = \Sigma \uplus A$;

- $\mathcal{P}'_a = \mathcal{L}_{(0,1]}$ for each $a \in A$ and $\mathcal{P}'_\sigma = \mathcal{P}_\sigma$ for each $\sigma \in \Sigma$;
- $\Xi = S \times (\mathbb{R}_0^+)^A$;
- $T_b^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega) = \begin{cases} X_b|_{x_b}(\omega_b) & \text{pt } b \in \Gamma^s \\ +\infty & \text{otherwise} \end{cases}$ for each $\langle s,\{x_a\}_{a\in A}\rangle \in \Xi$ and $b \in A$;
- $\delta^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega)(t) = \begin{cases} \langle s, \{x_a + t\}_{a\in A}\rangle & \text{pt } t < \mu^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega) \\ \langle \Delta^s_{\alpha^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega)}(\omega_\Sigma), y^{\langle s,\{x_a\}_{a\in A}\rangle}(\omega_A)\rangle & \text{otherwise} \end{cases}$
  for each $\langle s,\{x_a\}_{a\in A}\rangle \in \Xi$.

The main idea behind extracting an rta from an sta is that, both should produce the same 'behavior'. That is, if for some outcome a set of actions occurs in the sta it should also occur in the corresponding extracted rta.

The set of random sources has to be increased with the set of actions, since in the rta each of them will be needed to describe the T's. Observe that in sta's, the random sources in Σ are used only to describe the Δ's. The configurations indicate the state and the aging time for each action. The T's are obtained from X's having in mind the aging times and the enabled actions. Finally, δ is obtained either from time passing without any action occurring or by Δ when a set of actions occurs.

2.4 Deterministic transition rta's

Some special cases of rta's are of great interest in applications; for instance, in stochastic Petri nets the random time of an action to occur depends on previous times of occurrence of actions. These dependencies are expressed using *preemptive policies* [16], like preemptive repeat different and preemptive resume. We propose a stochastic version of the preemptive resume policy (that encompasses the non-stochastic version):

Definition 11 An rta is said to be an *RTC-rta* iff, for every $\xi \in \Xi$, $t \in \mathbb{R}^+$ and $x_a \in \mathbb{R}^+$,

$$\{\langle \omega_1, \omega_2\rangle : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t, \bigwedge_{a\in A} T_a^{\delta^\xi(\omega_1)(t)}(\omega_2) \leq x_a\}$$

is a measurable set, and

$$\mathcal{P}^{\bullet 2}(\{\langle \omega_1, \omega_2\rangle : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t, \bigwedge_{a\in A} T_a^{\delta^\xi(\omega_1)(t)}(\omega_2) \leq x_a\})$$

$$= \mathcal{P}^\bullet(\{\omega : \bigwedge_{a\in A} t < T_a^\xi(\omega) \leq x_a + t\}).$$

Automata in the class above fulfill a *residual time constraint* (RTC). This property asserts that if from a configuration ξ no action occurs in t units of time then the distribution of the random times from the random configuration reached (from ξ in time t) is the residual time distribution of the random times from ξ. The non-stochastic version of RTC (which corresponds to preemptive resume policy) is the special case where

$$T_a^{\delta^\xi(\omega_1)(t)}(\omega_2) = T_a^\xi(\omega_1) - t \text{ for all } \omega_2 \in \Omega^\bullet \text{ whenever } t < \mu^\xi(\omega_1).$$

Observe that the random times from configuration $\delta^\xi(\omega_1)(t)$ are constants in this case. Another special class of rta's that is interesting for applications is the following:

Definition 12 An rta is said to be a *DTC-rta* (Deterministic Transition Constrained) iff, for every $\xi \in \Xi$ and $\omega_1, \omega_2 \in \Omega^\bullet$:

(1) if $t < \min\{\mu^\xi(\omega_1), \mu^\xi(\omega_2)\}$ then

$$\delta^\xi(\omega_1)(t) = \delta^\xi(\omega_2)(t);$$

(2) if $\mu^\xi(\omega_1) = \mu^\xi(\omega_2)$ and $\alpha^\xi(\omega_1)(\mu^\xi(\omega_1)) = \alpha^\xi(\omega_2)(\mu^\xi(\omega_2))$ then

$$\delta^\xi(\omega_1)(\mu^\xi(\omega_1)) = \delta^\xi(\omega_2)(\mu^\xi(\omega_2)).$$

For these automata, the transitions from a given configuration are totally determined by the set of winning actions and the winning time. The first condition is a *deterministic transition before winning time constraint* (DTBW) which asserts that the configuration evolution before the winning time is determined by the cut time and the starting configuration. The other condition is a *deterministic transition at winning time constraint* (DTAW) which imposes that the configuration reached at the winning time only depends on the set of winning actions, the winning time and the starting configuration.

Remark 13 The rta's presented in Example 5 and Example 6, modeling an ideal semaphore and a semaphore, are DTC-rta's that satisfy the RTC condition.

An rta extracted from an sta, as shown in Proposition 10, satisfies the RTC condition but does not verify the DTC constraint unless the DTAW condition is verified, which happens only if Δ is non-probabilistic (that is, deterministic). This is always true whenever $\mathcal{P}^\bullet = \prod_{\sigma\in\Sigma} \mathcal{P}_\sigma$ is the trivial probability space (i.e., the probability space with only one outcome, as is the case in Example 9).

In the case of a DTC-rta, the residual time constraint can be stated in a much clear way:

Proposition 14 A DTC-rta is an RTC-rta iff for every $\xi \in \Xi$, $\omega \in \Omega^\bullet$ and $t \in (0, \mu^\xi(\omega))$,

$$P^\bullet(\bigwedge_{a\in A} T_a^{\delta^\xi(\omega)(t)} \leq x_a) = P^\bullet(\bigwedge_{a\in A} T_a^\xi \leq x_a + t \mid \bigwedge_{a\in A} t < T_a^\xi)$$

whenever $P^\bullet(\bigwedge_{a\in A} t < T_a^\xi) > 0$.

PROOF. Assume that DTBW holds. Observe that the previous condition can be written in the following way:

$$P^\bullet(\bigwedge_{a\in A} T_a^{\delta^\xi(\omega)(t)} \leq x_a) \times P^\bullet(\bigwedge_{a\in A} t < T_a^\xi) = P^\bullet(\bigwedge_{a\in A} t < T_a^\xi \leq x_a + t)$$

whenever $P^\bullet(\bigwedge_{a\in A} t < T_a^\xi) > 0$.
Note that the following set is measurable:

$$\begin{aligned}\{\langle\omega_1, \omega_2\rangle : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t, \bigwedge_{a\in A} T_a^{\delta^\xi(\omega_1)(t)}(\omega_2) \leq x_a\} \\ = \{\omega_1 : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t\} \times \{\omega_2 : \bigwedge_{a\in A} T_a^{\delta^\xi(t)}(\omega_2) \leq x_a\}\end{aligned}$$

where $\delta^\xi(t) = \delta^\xi(\omega)(t)$ whenever $t < \mu^\xi(\omega)$. We only consider the case where $P^\bullet(\{\omega_1 : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t\}) > 0$ (otherwise the equality holds with both sides equal to 0). Observe that if $\bigwedge_{a\in A} T_a^\xi(\omega_1) > t$ then $\mu^\xi(\omega_1) > t$. So,

$$\begin{aligned}&P^{\bullet 2}(\{\langle\omega_1, \omega_2\rangle : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t, \bigwedge_{a\in A} T_a^{\delta^\xi(\omega_1)(t)}(\omega_2) \leq x_a\}) \\ &\quad = P^{\bullet 2}(\{\omega_1 : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t\} \times \{\omega_2 : \bigwedge_{a\in A} T_a^{\delta^\xi(t)}(\omega_2) \leq x_a\}) \\ &\quad = P^\bullet(\{\omega_1 : \bigwedge_{a\in A} T_a^\xi(\omega_1) > t\}) \times P^\bullet(\{\omega_2 : \bigwedge_{a\in A} T_a^{\delta^\xi(t)}(\omega_2) \leq x_a\}) \\ &\quad = P^\bullet(\{\omega : \bigwedge_{a\in A} t < T_a^\xi(\omega) \leq x_a + t\}). \qquad \square\end{aligned}$$

Observe that to obtain the above result we only require that the rta satisfies the DTBW condition, which is the case for DTC-rta's.

We now turn our attention to the problem of setting up the appropriate category of rta's. To this end, we use the large subcategory $\mathbf{Rel}^-$ of $\mathbf{Rel}$ [3] constituted by all *exhaustive* relations, that is, relations $R \subseteq A \times A'$ such that for every $a' \in A'$ there is an $a \in A$ with $\langle a, a' \rangle \in R$. It is easy to check that $\mathbf{Rel}^-$ has the same products as $\mathbf{Rel}$ [4] . For the sake of simplicity we present the morphisms in $\mathbf{Rel}^-$, i.e., exhaustive relations $R \subseteq A \times A'$ as maps $h_R : A \to \wp A'$ where $h_R(a) = \{a' \in A' : \langle a, a' \rangle \in R\}$.

Definition 15 An *rta morphism* $h : m \to m'$ is a triple $\langle h^s, h^a, h^c \rangle$ where:

- $h^s : \Sigma' \to \Sigma$ in $\mathbf{Set}$;
- $h^a : A \to \wp A'$ in $\mathbf{Rel}^-$;
- $h^c : \Xi \to \Xi'$ in $\mathbf{Set}_*$;

such that:

(1) $\mathcal{P}'_{\sigma'} = \mathcal{P}_{h^s(\sigma')}$;
(2) for every $a' \in h^a(a)$:
  - $T^\xi_a(\omega) \leq {T'}^{h^c(\xi)}_{a'}(\omega \circ h^s)$ where $(\omega \circ h^s)_{\sigma'} = \omega_{h^s(\sigma')}$;

(3) $h^c(\delta^\xi(\omega)(t)) = {\delta'}^{h^c(\xi)}(\omega \circ h^s)(t)$ for every $t \in (0, \mu^\xi(\omega)]$.

This notion of morphism is different from the one considered in [26,31], where a morphism included a 'map' between probability spaces. Moreover, in [26,31], random variables defined over isomorphic probability spaces would have the same distribution function. For instance, two fair coins were indistinguishable. With the proposed notion of morphism, we are able to distinguish two fair coins, based on the fact that an outcome tails in one does not imply an outcome tails in the other. Therefore, an isomorphism between rta's will imply that the same sequence of outcomes will occur in both rta's. This setting is stronger than the approach in [26,31], since it implies equality at the distribution level as well. Note that for aggregation and interconnection, to be discussed latter on, we want to relate sequences of outcomes in the components with sequences of outcomes in the composite automaton, having in mind that whenever an outcome occurs in the component it also occurs in the composite.

In order to understand better this issue, consider the following example: we have two players, A and B, and assume that when tossing a fair coin A wins

[3] The category $\mathbf{Rel}$ is the category whose objects are sets and whose morphisms are relations.

[4] Products in $\mathbf{Rel}$ are as follows: $A' \otimes A''$ is (up to isomorphism) the triple composed of the object $A' \uplus A''$, the projection $\pi' = {\iota'}^{-1}$, and the projection $\pi'' = {\iota''}^{-1}$, where $\iota' : A' \hookrightarrow A' \uplus A''$ and $\iota'' : A'' \hookrightarrow A' \uplus A''$.

a point if the result is tails and B wins a point otherwise. We should consider only one random source $\langle\{t,h\},\wp\{t,h\},p(t)=p(h)=\frac{1}{2}\rangle$, and the random variables X_A and X_B denoting the amount won by players A and B after one toss. The random variables X_A and X_B have the same distribution, that is $P(\{\omega\in\{t,h\}:X_A(\omega)=1\})=P(\{\omega\in\{t,h\}:X_B(\omega)=1\})$. However $\{\omega\in\{t,h\}:X_A(\omega)=1\}\neq\{\omega\in\{t,h\}:X_B(\omega)=1\}$, and moreover, $\{\omega\in\{t,h\}:X_A(\omega)=1\}\cap\{\omega\in\{t,h\}:X_B(\omega)=1\}=\emptyset$. In the former setting, X_A and X_B are 'isomorphic' via $g(t)=h$ and $g(h)=t$. In the present approach they are not related.

In addition, there is a probabilistic technical problem when adopting the purely distribution point of view (like in [10]). Knowing the distributions of two random quantities does not imply any specific joint distribution unless independence is assumed, which is not the case when considering interconnection. With our definition of morphism, we were able to overcome this problem.

We aim at obtaining the 'aggregation' (parallel composition) of two given rta's as their product. Furthermore, we aim at explaining interconnection via 'action calling' as a cartesian lifting. This objective explains the choice of the 'multimap' on actions: each action is mapped to the set of actions that call it. It also explains the condition on the random times: the time of the called action should be less than or equal to the times of those that call it. Relaxing the strict condition on the probability spaces is possible (for details see [25] where a comparison is made to the precategorical approach), but not necessary in this paper. The condition on the transition functions is obvious. However it is well defined only if the winning time of the domain rta is less or equal than the winning time of the codomain rta. This last requirement is guaranteed by the exhaustiveness of the multimap and Condition 2, as stated below.

Proposition 16 If $h:m\to m'$, then $\mu^{\xi}(\omega)\leq {\mu'}^{h^c(\xi)}(\omega\circ h^s)$.

Randomly timed automata and their morphisms constitute the category **Rta**. DTC-rta's and their morphisms constitute the full subcategory **dRta** of **Rta**.

3 Minimization

We now extend to rta's the classical notion of minimization by merging equivalent configurations. As usual, we start with the notion of sober automaton.

Definition 17 An rta is said to be *sober* iff every $\xi\in\Xi$ is *accessible*, that is, there are $\vec{\omega}$ and $\vec{t}$ such that $\delta^{*\xi_0}(\vec{\omega})(\vec{t})=\xi$.

We denote by $Sob(m)$ the rta obtained from m by removing the non accessible configurations. Equivalence is as expected:

Definition 18 Two configurations ξ_1, ξ_2 are said to be *equivalent* (written $\xi_1 \approx \xi_2$) iff

$$T^{\delta^{*\xi_1}(\vec{\omega})(\vec{t})} = T^{\delta^{*\xi_2}(\vec{\omega})(\vec{t})}$$

for every outcome sequence $\vec{\omega}$ and admissible cut sequence $\vec{t}$.

The relation is imposed on the random times, but as the next result shows, equivalence is propagated by transitions as it should:

Proposition 19 If $\xi_1 \approx \xi_2$, then:

- $\mu^{\xi_1} = \mu^{\xi_2}$;
- $\alpha^{\xi_1} = \alpha^{\xi_2}$;
- $\delta^{\xi_1}(\omega)(t) \approx \delta^{\xi_2}(\omega)(t)$;
- $\delta^{*\xi_1}(\vec{\omega})(\vec{t}\,) \approx \delta^{*\xi_2}(\vec{\omega})(\vec{t}\,)$.

PROOF.
(i) Take $\hat{\omega} = \epsilon$ and $\hat{t} = \epsilon$. Then, because $\xi_1 \approx \xi_2$, we have $T^{\xi_1} = T^{\xi_2}$ and so $T_a^{\xi_1} = T_a^{\xi_2}$ for every $a \in A$. Therefore, $\mu^{\xi_1} = \mu^{\xi_2}$ and $\alpha^{\xi_1} = \alpha^{\xi_2}$.
(ii) Furthermore, observe that

$$\begin{aligned} T^{\delta^{*\delta^{\xi_1}(\omega)(t)}(\vec{\omega})(\vec{t})} &= T^{\delta^{*\xi_1}(\omega\vec{\omega})(t\vec{t})} \\ &= T^{\delta^{*\xi_2}(\omega\vec{\omega})(t\vec{t})} \\ &= T^{\delta^{*\delta^{\xi_2}(\omega)(t)}(\vec{\omega})(\vec{t})}. \end{aligned}$$

Thus, we also have $\delta^{\xi_1}(\omega)(t) \approx \delta^{\xi_2}(\omega)(t)$ whenever $\xi_1 \approx \xi_2$. Furthermore, the last result follows by induction. □

Definition 20 A sober rta is said to be *minimal* iff its equivalence $\approx$ is the diagonal relation.

As usual, we only consider minimization of sober automata. Given a sober rta m, we denote by $Min(m)$ the quotient rta $\langle \Sigma, \mathcal{P}, A, \Xi/\approx, T/\approx, \delta/\approx \rangle$. Sober rta's and their morphisms constitute the full subcategory **sRta** of **Rta**. Furthermore, sober rta's over $\Sigma, \mathcal{P}, A$ and their morphisms such that:

- both h^s and h^a are identities;
- $T_a^{\xi}(\omega) = {T'}_a^{h^c(\xi)}(\omega)$;

constitute the category **sRta**$(\Sigma, \mathcal{P}, A)$. The full subcategory of **sRta**$(\Sigma, \mathcal{P}, A)$ constituted by all minimal rta's is denoted by **mRta**$(\Sigma, \mathcal{P}, A)$.

Theorem 21 The map *Sob* on rta's extends to a right adjoint functor from **Rta** to **sRta** with the inclusion as left adjoint. The map *Min* extends to a left adjoint functor from $\mathbf{sRta}(\Sigma, \mathcal{P}, A)$ to $\mathbf{mRta}(\Sigma, \mathcal{P}, A)$ with the inclusion as right adjoint. Therefore, **sRta** is a co-reflective subcategory of **Rta** and $\mathbf{mRta}(\Sigma, \mathcal{P}, A)$ is a reflective subcategory of $\mathbf{sRta}(\Sigma, \mathcal{P}, A)$.

PROOF.
(i) Observe that, given a morphism $h : m \to m'$ in **Rta**, we have $h^c(\delta^{*\xi}(\vec{\omega})(\vec{t})) = \delta'^{*h^c(\xi)}(\vec{\omega} \circ h^s)(\vec{t})$ by induction on the size of $\vec{\omega}$. Therefore $h^c(\xi)$ is accessible providing that ξ is accessible. For each rta m, take as co-reflection of m the inclusion $\iota_m : Sob(m) \hookrightarrow m$. For every rta morphism $h : m' \to m$ where m' is sober, $\tilde{h} = h$ is the unique morphism from m' to $Sob(m)$ making the diagram to commute (note that $\tilde{h} : m' \to Sob(m)$ is a morphism because h preserves accessible configurations).

The result follows by noticing that morphisms preserve accessible configurations.
(ii) Note that, given a morphism $h : m \to m'$ in $\mathbf{sRta}(\Sigma, \mathcal{P}, A)$, we have $h^c(\xi_1) \approx h^c(\xi_2)$ whenever $\xi_1 \approx \xi_2$. For each sober rta m take as reflection of m the natural map $\eta_m : m \to Min(m)$ where $\eta_m(\xi) = [\xi]_\approx$. The reflection is clearly a morphism by definition of $\approx$. For every srta morphism $h : m \to m'$ where m' is minimal, $\tilde{h} : Min(m) \to m'$ where $\tilde{h}^c([\xi]_\approx) = h^c(\xi)$ is the unique morphism that makes the diagram to commute (note that if $\xi_1 \approx \xi_2$ then $\tilde{h}^c(\xi_1) \approx \tilde{h}^c(\xi_2)$, and since m' is minimal, then $\tilde{h}^c(\xi_1) = \tilde{h}^c(\xi_2)$, and so $\tilde{h}$ is well defined). □

As in the classical case, we do minimization fiberwise [18]. This is so because we want to find the minimal realization for a given Σ, $\mathcal{P}$ and A.

4 Aggregation, encapsulation and interconnection

Within the category **Rta** we now define as universal constructions three mechanisms for building new rta's from given rta's, adapting to rta's the approach advocated in [35,32].

4.1 Aggregation

We start with the simplest form of combination of rta's: *aggregation* as the product of two rta's. This construction is easily extended to the product of a finite family of automata, but for the sake of brevity of presentation we refrain from considering the general case. Aggregation corresponds to putting together the given automata without any form of interaction (parallel composition).

Proposition 22 Let m' and m'' be rta's. The product of m' and m'' is the rta $m' \otimes m'' = \langle \Sigma' \uplus \Sigma'', [\mathcal{P}', \mathcal{P}''], A' \uplus A'', \Xi' \times \Xi'', T, \delta \rangle$ where

- $[\mathcal{P}', \mathcal{P}'']$ is such that $[\mathcal{P}', \mathcal{P}'']_{\iota^{s'}(\sigma')} = \mathcal{P}'_{\sigma'}$ and $[\mathcal{P}', \mathcal{P}'']_{\iota^{s''}(\sigma'')} = \mathcal{P}''_{\sigma''}$;
- $T^{\langle \xi', \xi'' \rangle} = [T'^{\xi'}, T''^{\xi''}]$ where $[T'^{\xi'}, T''^{\xi''}]_{\iota^{a'}(a')} = T'^{\xi'}_{a'}$ and $[T'^{\xi'}, T''^{\xi''}]_{\iota^{a''}(a'')} = T'^{\xi''}_{a''}$;
- $\delta^{\langle \xi', \xi'' \rangle}(\langle \omega', \omega'' \rangle)(t) = \langle \delta'^{\xi'}(\omega')(t), \delta''^{\xi''}(\omega'')(t) \rangle$ for $t \in (0, \mu^{\langle \xi', \xi'' \rangle}(\langle \omega', \omega'' \rangle)]$;

endowed with the projections:

- $\langle \iota^{s'} : \Sigma' \to \Sigma' \uplus \Sigma'', \pi^{a'} : A' \uplus A'' \to \wp A', \pi^{c'} : \Xi' \times \Xi'' \to \Xi' \rangle$;
- $\langle \iota^{s''} : \Sigma'' \to \Sigma' \uplus \Sigma'', \pi^{a''} : A' \uplus A'' \to \wp A'', \pi^{c''} : \Xi' \times \Xi'' \to \Xi'' \rangle$;

where $\pi^{a'} = \iota^{a'-1}$ and $\pi^{a''} = \iota^{a''-1}$ with $\iota^{a'} : A' \hookrightarrow A' \uplus A''$ and $\iota^{a''} : A'' \hookrightarrow A' \uplus A''$.

PROOF.
(i) First observe that $\mu^{\langle \xi', \xi'' \rangle}(\langle \omega', \omega'' \rangle) = \min\{\mu^{\xi'}(\omega'), \mu^{\xi''}(\omega'')\}$. Hence, δ is well defined.
(ii) We show that the projections are rta-morphisms, assuming without loss of generality that $\pi'(a) = \{a'\}$:

$$\begin{aligned} T_a^{\langle \xi', \xi'' \rangle}(\langle \omega', \omega'' \rangle) &= [T^{\xi'}, T^{\xi''}]_a(\langle \omega', \omega'' \rangle) \\ &= T_{a'}^{\xi'}(\omega') \\ &= T_{a'}^{\xi'}(\langle \omega', \omega'' \rangle \circ \iota'). \end{aligned}$$

(iii) Finally, we check the universal property. Given $h' : m''' \to m'$ and $h'' : m''' \to m''$, consider $h = \langle [h^s, h''^s], \lambda a. h'^a(a) \cup h''^a(a), \langle h'^c, h''^c \rangle \rangle$. We show that h is an rta-morphism assuming without loss of generality that $\pi'(a) = \{a'\}$. Observe that

$$\begin{aligned} T'^{h^{c'}(\xi)}_{a'}(\omega''' \circ h^{s'}) &= [T'^{h^{c'}(\xi''')}, T''^{h^{c''}(\xi''')}]_a(\langle \omega''' \circ h'^s, \omega''' \circ h''^s \rangle) \\ &= [T'^{h^{c'}(\xi''')}, T''^{h^{c''}(\xi''')}]_a(\omega''' \circ h^s) \\ &= T_a^{h^c(\xi''')}(\omega''' \circ h^s). \end{aligned}$$

Hence, $T^{\xi'''}_{a'''}(\omega''') \leq T_a^{h^c(\xi''')}(\omega''' \circ h^s)$. $\square$

We delay the illustration of this construction until the end of this section where we present an example of interconnection by action calling built upon an aggregation. We conclude this subsection with some closure results concerning the classes of rta's introduced in Section 2.

Proposition 23 The product of two DTC-rta's is a DTC-rta.

PROOF. Assume that $t < \min\{\mu^{\langle\xi',\xi''\rangle}(\langle\omega_1',\omega_1''\rangle), \mu^{\langle\xi',\xi''\rangle}(\langle\omega_2',\omega_2''\rangle)\}$. Therefore, $t < \min\{\mu^{\xi'}(\omega_1'), \mu^{\xi''}(\omega_1''), \mu^{\xi'}(\omega_2'), \mu^{\xi''}(\omega_2'')\}$. Then:

$$\begin{aligned}\delta^{\langle\xi',\xi''\rangle}(\langle\omega_1',\omega_1''\rangle)(t) &= \langle\delta'^{\xi'}(\omega_1')(t), \delta''^{\xi''}(\omega_1'')(t)\rangle \\ &= \langle\delta'^{\xi'}(\omega_2')(t), \delta''^{\xi''}(\omega_2'')(t)\rangle \\ &= \delta^{\langle\xi',\xi''\rangle}(\langle\omega_2',\omega_2''\rangle)(t).\end{aligned}$$

Hence, the class of DTC-rta's is closed under products. $\square$

Proposition 24 The product of two DTC-RTC-rta's is an RTC-rta.

PROOF. Let $A = \{\iota'(a_1'), \ldots, \iota'(a_{n'}'), \iota''(a_1''), \ldots, \iota''(a_{n''}'')\}$ and assume that:

$$0 < [P', P'']^\bullet(\bigwedge_{a\in A} t < T_a^{\langle\xi',\xi''\rangle}).$$

First, observe that if $0 \leq y_a < x_a \leq +\infty$ for all $a \in A$ then

$$\begin{aligned}&[P', P'']^\bullet(\bigwedge_{a\in A} y_a < T_a^{\langle\xi',\xi''\rangle} \leq x_a) \\ &= [P', P'']^\bullet(\bigwedge_{k'=1}^{n'} y_{\iota'(a_{k'}')} < T_{\iota'(a_{k'}')}^{\langle\xi',\xi''\rangle} \leq x_{\iota'(a_{k'}')} \wedge \bigwedge_{k''=1}^{n''} y_{\iota''(a_{k''}'')} < T_{\iota''(a_{k''}'')}^{\langle\xi',\xi''\rangle} \leq x_{\iota''(a_{k''}'')}) \\ &= [P', P'']^\bullet(\bigwedge_{k'=1}^{n'} y_{\iota'(a_{k'}')} < T_{\iota'(a_{k'}')}^{\langle\xi',\xi''\rangle} \leq x_{\iota'(a_{k'}')}) \\ &\qquad \times [P', P'']^\bullet(\bigwedge_{k''=1}^{n''} y_{\iota''(a_{k''}'')} < T_{\iota''(a_{k''}'')}^{\langle\xi',\xi''\rangle} \leq x_{\iota''(a_{k''}'')}) \\ &= P'^\bullet(\bigwedge_{k'=1}^{n'} y_{\iota'(a_{k'}')} < T'^{\xi'}_{a_{k'}'} \leq x_{\iota'(a_{k'}')}) \times P''^\bullet(\bigwedge_{k''=1}^{n''} y_{\iota''(a_{k''}'')} < T''^{\xi''}_{a_{k''}''} \leq x_{\iota''(a_{k''}'')}).\end{aligned}$$

In particular, with $y_a = t$ and $x_a = +\infty$ for all $a \in A$, we conclude that:

$$\begin{aligned}0 &< P'^\bullet(\bigwedge_{k'=1}^{n'} t < T'^{\xi'}_{a_{k'}'}); \\ 0 &< P''^\bullet(\bigwedge_{k''=1}^{n''} t < T''^{\xi''}_{a_{k''}''}).\end{aligned}$$

Furthermore by choosing $\langle\omega',\omega''\rangle$ such that:

$$t < \inf\{T_a^{\langle\xi',\xi''\rangle}(\langle\omega',\omega''\rangle) : a \in A\}$$

we also have

$$t < \inf\{{T'}^{\xi'}_{\iota'(a'_{k'})}(\omega') : k' \in \{1,\dots,n'\}\};$$
$$t < \inf\{{T''}^{\xi''}_{\iota''(a''_{k''})}(\omega'') : k'' \in \{1,\dots,n''\}\}.$$

Therefore:

$$\begin{aligned}
&[P',P'']^\bullet(\bigwedge_{a\in A} T_a^{\delta^\xi(\langle\omega',\omega''\rangle)(t)} \leq x_a)\\
&= [P',P'']^\bullet(\bigwedge_{k'=1}^{n'} T_{\iota'(a'_{k'})}^{\delta^{\langle\xi',\xi''\rangle}(\langle\omega',\omega''\rangle)(t)} \leq x_{\iota'(a'_{k'})}\\
&\qquad \wedge \bigwedge_{k''=1}^{n''} T_{\iota''(a''_{k''})}^{\delta^{\langle\xi',\xi''\rangle}(\langle\omega',\omega''\rangle)(t)} \leq x_{\iota''(a''_{k''})})\\
&= [P',P'']^\bullet(\bigwedge_{k'=1}^{n'} T_{\iota'(a'_{k'})}^{\delta^{\langle\xi',\xi''\rangle}(\langle\omega',\omega''\rangle)(t)} \leq x_{\iota'(a'_{k'})})\\
&\quad \times [P',P'']^\bullet(\bigwedge_{k''=1}^{n''} T_{\iota''(a''_{k''})}^{\delta^{\langle\xi',\xi''\rangle}(\langle\omega',\omega''\rangle)(t)} \leq x_{\iota''(a''_{k''})})\\
&= {P'}^\bullet(\bigwedge_{k'=1}^{n'} {T'}_{a'_{k'}}^{{\delta'}^{\xi'}(\omega')(t)} \leq x_{\iota'(a'_{k'})}) \times {P''}^\bullet(\bigwedge_{k''=1}^{n''} {T''}_{a''_{k''}}^{{\delta''}^{\xi''}(\omega'')(t)} \leq x_{\iota''(a''_{k''})})\\
&= {P'}^\bullet(\bigwedge_{k'=1}^{n'} {T'}^{\xi'}_{a'_{k'}} \leq x_{\iota'(a'_{k'})} + t \mid \bigwedge_{k'=1}^{n'} t < {T'}^{\xi'}_{a'_{k'}})\\
&\quad \times {P''}^\bullet(\bigwedge_{k''=1}^{n''} {T''}^{\xi''}_{a''_{k''}} \leq x_{\iota''(a''_{k''})} + t \mid \bigwedge_{k''=1}^{n''} t < {T''}^{\xi''}_{a''_{k''}})\\
&= \frac{{P'}^\bullet(\bigwedge_{k'=1}^{n'} t < {T'}^{\xi'}_{a'_{k'}} \leq x_{\iota'(a'_{k'})} + t)}{{P'}^\bullet(\bigwedge_{k'=1}^{n'} t < {T'}^{\xi'}_{a'_{k'}})}\\
&\quad \times \frac{{P''}^\bullet(\bigwedge_{k''=1}^{n''} t < {T''}^{\xi''}_{a''_{k''}} \leq x_{\iota''(a''_{k''})} + t)}{{P''}^\bullet(\bigwedge_{k''=1}^{n''} t < {T''}^{\xi''}_{a''_{k''}})}\\
&= \frac{[P',P'']^\bullet(\bigwedge_{a\in A} t < T_a^{\langle\xi',\xi''\rangle} \leq x_a + t)}{[P',P'']^\bullet(\bigwedge_{a\in A} t < T_a^{\langle\xi',\xi''\rangle})}\\
&= [P',P'']^\bullet(\bigwedge_{a\in A} T_a^{\langle\xi',\xi''\rangle} \leq x_a + t \mid \bigwedge_{a\in A} t < T_a^{\langle\xi',\xi''\rangle}). \qquad \square
\end{aligned}$$

Another interesting construction corresponds to hiding some actions from a given rta. To present this combination, we shall need the forgetful functor Act from **Rta** to $\mathbf{Rel}^-$ that extracts the alphabets of actions and their maps from rta's and their morphisms. This construction appears as a co-cartesian lifting by Act: given an rta m and an action morphism h^a from the alphabet A of m to the new alphabet $A' \subset A$ such that $h(a) = \emptyset$ for every $a \notin A'$, the resulting rta m' is calculated by lifting h^a to **Rta** by Act. In the resulting rta m', the actions of m that we want to hide (the elements of A not in A') are of course omitted. In general, we have:

Proposition 25 Let $m = \langle \Sigma, \mathcal{P}, A, \Xi, T, \delta \rangle$ be an rta and $h^a : A \to A'$ a morphism in $\mathbf{Rel}^-$ such that:

(1) $h^a(a) \cap h^a(b) = \emptyset$ for every $a \neq b \in A$;
(2) If $h^a(a) = \emptyset$, then for every $\omega \in \Omega^\bullet$ there is a $b \in A$ such that:
- $h^a(b) \neq \emptyset$;
- $T^\xi_a(\omega) \geq T^\xi_b(\omega)$.

Then, the co-cartesian lifting of h^a by Act on m is the rta morphism

$$\langle \mathrm{id}_\Sigma, h^a, \mathrm{id}_\Xi \rangle : m \to m'$$

with $m' = \langle \Sigma, \mathcal{P}, A', \Xi, T', \delta \rangle$ where ${T'}^\xi_{a'}(\omega) = T^\xi_{\iota a.\ a' \in h^a(a)}(\omega)$.

PROOF.
(i) Observe that ${\mu'}^\xi(\omega) = \mu^\xi(\omega)$ by the second hypothesis and, hence, δ has the right domain of definition in m'. Furthermore, it is straightforward to verify that $\langle \mathrm{id}_\Sigma, h^a, \mathrm{id}_\Xi \rangle$ is an rta morphism.
(ii) It remains to prove the universal property. Let $g^a : A' \to A''$ be a morphism in $\mathbf{Rel}^-$ and $f : m \to m''$ a morphism in **Rta** such that $g^a \circ h^a = f^a$. Again, it is straightforward to verify that $g = \langle f^s, g^a, f^c \rangle$ is an rta morphism. Uniqueness of g is straightforward. □

Observe that exhaustiveness and Condition (1) amount to requiring that the inverse of h^a is a map.

We denote by $h^a(m)$ the codomain of the co-cartesian lifting of h^a by Act on m. Note that Condition (2) above on the action morphisms for the existence of the co-cartesian lifting is very restrictive. It means that we can only hide an action that 'entails' another action not to be hidden. For this reason, such hiding is not very useful in practice (but further comments will be made later on with respect to 'calling'). We presented this notion of hiding just because it shall be used in the next subsection as an auxiliary tool. More precisely, we shall need the following closure results about hiding as a co-cartesian lifting:

Proposition 26 The codomain of the co-cartesian lifting on a given DTC-rta is an DTC-rta.

PROOF. Straightforward from the definition. □

Proposition 27 The codomain of the co-cartesian lifting on a given RTC-DTC-rta is an RTC-rta.

PROOF. Assume that $P^\bullet(\bigwedge_{a'\in A'} {T'}^{\xi}_{a'} > t) > 0$ and $t < \mu^\xi(\omega)$. Then:

$$
\begin{aligned}
&P^\bullet(\bigwedge_{a'\in A'} {T'}^{\delta^\xi(\omega)(t)}_{a'} \leq x_{a'}) \\
&\quad = P^\bullet(\bigwedge_{a'\in A'} T^{\delta^\xi(\omega)(t)}_{\iota a.\, a'\in h^a(a)} \leq x_{a'}) \\
&\quad = P^\bullet(\bigwedge_{a'\in A'} T^{\delta^\xi(\omega)(t)}_{\iota a.\, a'\in h^a(a)} \leq x_{a'} \wedge \bigwedge_{b\in\{b\in A : h(b)=\emptyset\}} T^{\delta^\xi(\omega)(t)}_{b} \leq +\infty) \\
&\quad = P^\bullet(\bigwedge_{a'\in A'} T^{\xi}_{\iota a.\, a'\in h^a(a)} \leq x_{a'} + t \wedge \bigwedge_{b\in\{b\in A : h(b)=\emptyset\}} T^{\xi}_{b} \leq +\infty \mid \\
&\qquad\quad \bigwedge_{a'\in A'} T^{\xi}_{\iota a.\, a'\in h^a(a)} > t \wedge \bigwedge_{b\in\{b\in A : h(b)=\emptyset\}} T^{\xi}_{b} > t) \\
&\quad = P^\bullet(\bigwedge_{a'\in A'} T^{\xi}_{\iota a.\, a'\in h^a(a)} \leq x_{a'} + t \mid \bigwedge_{a'\in A'} T^{\xi}_{\iota a.\, a'\in h^a(a)} > t) \\
&\quad = P^\bullet(\bigwedge_{a'\in A'} {T'}^{\xi}_{a'} \leq x_{a'} + t \mid \bigwedge_{a'\in A'} {T'}^{\xi}_{a'} > t). \qquad \square
\end{aligned}
$$

4.3 Interconnection

The most interesting form of combination of rta's involves some interaction between them. We achieve this by first obtaining their product (aggregation without any interaction) and then by imposing the envisaged interaction by a cartesian lifting of an appropriate morphism. An especially interesting form of interconnection is known as calling and will be introduced in the next subsection. First, we state the general results about the cartesian lifting:

Proposition 28 Let $m' = \langle \Sigma', \mathcal{P}', A', \Xi', T', \delta' \rangle$ be an rta and $h^a : A \to A'$ a morphism in $\mathbf{Rel}^-$. Then, the cartesian lifting of h^a by *Act* on m' is the rta morphism

$$\langle \mathrm{id}_{\Sigma'}, h^a, \mathrm{id}_{\Xi'} \rangle : m \to m'$$

with $m = \langle \Sigma', \mathcal{P}', A, \Xi', T, \delta' \rangle$ where $T^{\xi}_{a}(\omega) = \inf_{a' \in h^a(a)} T'^{\xi}_{a'}(\omega)$.

PROOF.
(i) First, observe that $\mu^{\xi'}(\omega') = \mu'^{\xi'}(\omega')$ and, so, δ' is well defined in m.
(ii) It is straightforward to see that $\langle \mathrm{id}_{\Sigma'}, h^a, \mathrm{id}_{\Xi'} \rangle$ is a morphism.
(iii) Finally, we check the universal property. Let $g^a : A'' \to A$ be a morphism in $\mathbf{Rel}^-$ and $f : m'' \to m'$ a morphism in $\mathbf{Rta}$ such that $h^a \circ g^a = f^a$. We show that $g = \langle f^s, g^a, f^c \rangle$ is a morphism in $\mathbf{Rta}$. Let $a \in g^a(a'')$. Assume that $h^a(a) \neq \emptyset$. Then,

$$\begin{aligned}
T''^{\xi''}_{a''}(\omega'') &\leq \inf_{a' \in f^a(a'')} T'^{f^c(\xi'')}_{a'}(\omega'' \circ f^s) \\
&= \inf_{a' \in h^a(g^a(a''))} T'^{g^c(\xi'')}_{a'}(\omega'' \circ g^s) \\
&\leq \inf_{a' \in h^a(a)} T'^{g^c(\xi'')}_{a'}(\omega'' \circ g^s) \\
&= T^{g^c(\xi'')}_{a}(\omega'' \circ g^s).
\end{aligned}$$

When $h^a(a) = \emptyset$, it is trivial to see that g is a morphism. □

We denote by $h^{a-1}(m')$ the domain of the cartesian lifting of h^a by Act on m'.

Proposition 29 The domain of the cartesian lifting of $h : A \to A' \in \mathbf{Rel}^-$ on a given DTC-rta is a DTC-rta provided that for all $a' \in A'$ there is an $a \in A$ such that $h^a(a) = \{a'\}$.

PROOF. Straightforward from definition. □

Definition 30 A morphism $h : A \to A'$ in $\mathbf{Rel}^-$ is *simple* iff there exits $a \in A$ such that $h(a)$ has two elements and $h(b)$ is a singleton for all $b \in A \setminus \{a\}$.

In order to show the preservation of the RTC by cartesian lifting, it is important to observe that every morphism can be decomposed into simple morphisms and one hiding.

Lemma 31 Let m' be an rta such that $Act(m') = A'$ and $h^a : A \to A'$ be a morphism in $\mathbf{Rel}^-$ such that $h^a(a) \neq \emptyset$ for all $a \in A$. Then there exist $n \in \mathbb{N}$ and ${h^a}_1, \ldots, {h^a}_n, {h^a}_{n+1}$ such that ${h^a}_i$ is simple for $i \in \{1, ..., n\}$ and $h^{a-1}(m') = {h^a}_{n+1}(h^{a-1}_{n}(\ldots h^{a-1}_{1}(m') \ldots))$.

PROOF. Without loss of generality, we assume that $A = \{a_1, \ldots, a_n, a_{n+1}\}$, $A' = \{a'_1, \ldots, a'_n\}$, $h^a(a_k) = \{a'_k\}$ for $k \in \{1, \ldots, n\}$ and $h^a(a_{n+1}) = \{a'_i, a'_j, a'_p\}$, all other cases can be reduced, by factorization under composition, to this case. Then consider:

- $h^a{}_1 : \{b_1, \ldots, b_n, b_{n+1}\} \to A'$ where $h^a{}_1(b_k) = \{a'_k\}$ for $k \in \{1, \ldots, n\}$ and $h^a{}_1(b_{n+1}) = \{a'_i, a'_j\}$;
- $h^a{}_2 : \{b_1, \ldots, b_n, b_{n+1}, b_{n+2}\} \to \{b_1, \ldots, b_n, b_{n+1}\}$ where $h^a{}_2(b_k) = \{b_k\}$ for $k \in \{1, \ldots, n\}$, $h^a{}_2(b_{n+1}) = \{b_{n+1}, b_p\}$ and $h^a{}_2(b_{n+2}) = \{b_{n+1}\}$;
- $h^a{}_3 : \{b_1, \ldots, b_n, b_{n+1}, b_{n+2}\} \to A$ with $h^a{}_3(b_k) = \{a_k\}$ for $k \in \{1, \ldots, n+1\}$ and $h^a{}_3(b_{n+2}) = \emptyset$.

It is now straightforward to check that $h^{a-1}(m') = h^a{}_3(h^{a}{}_2^{-1}(h^{a}{}_1^{-1}(m')))$. $\square$

Proposition 32 Let $h : A \to A' \in \mathbf{Rel}^-$ be such that for all $a' \in A'$ there is an $a \in A$ such that $h^a(a) = \{a'\}$. The domain of the cartesian lifting of h on a given RTC-DTC-rta is an RTC-rta.

PROOF. We just consider the case where $h^a : A \to \wp A'$ is such that $h^a(a) \neq \emptyset$ for all $a \in A$. According to Proposition 27 and Lemma 31 it is enough to consider domains of cartesian lifting of simple morphisms. Assume that $A = \{a_1, ..., a_n, a_{n+1}\}$, $A' = \{a'_1, ..., a'_n\}$ and $h : A \to \wp A'$ is such that $h^a(a_k) = \{a'_k\}$ for $k = 1, ..., n$ and $h^a(a_{n+1}) = \{a'_i, a'_m\}$ for some $i, m \in \{1, ..., n\}$. Furthermore, suppose that $P'^{\bullet}(\bigwedge_{k=1,...,n+1} T_{a_k}^{\delta\xi'(\omega')(t)} > t) > 0$ and that $t < \mu^{\xi'}(\omega')$. Then

$$\begin{aligned}
&P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T_{a_k}^{\delta\xi'(\omega')(t)} \leq x_k \wedge T_{a_{n+1}}^{\delta\xi'(\omega')(t)} \leq x_{n+1}) \\
&\quad = P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T'^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_k \wedge \inf\{T'^{\delta\xi'(\omega')(t)}_{a'_i}, T'^{\delta\xi'(\omega')(t)}_{a'_m}\} \leq x_{n+1}).
\end{aligned}$$

1st case – $x_{n+1} > \max\{x_i, x_m\}$:

$$\begin{aligned}
&P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T'^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_k \wedge \inf\{T'^{\delta\xi'(\omega')(t)}_{a'_i}, T'^{\delta\xi'(\omega')(t)}_{a'_m}\} \leq x_{n+1}) \\
&\quad = P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T'^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_k) \\
&\quad = P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T'^{\xi'}_{a'_k} \leq x_k + t \mid \bigwedge_{k=1,\ldots,n} T'^{\xi'}_{a'_k} > t) \\
&\quad = P'^{\bullet}(\bigwedge_{k=1,\ldots,n} T'^{\xi'}_{a'_k} \leq x_k + t \wedge \inf\{T'^{\xi'}_{a'_i}, T'^{\xi'}_{a'_m}\} \leq x_{n+1} + t \\
&\qquad\qquad \mid \bigwedge_{k=1,\ldots,n} T^{\xi'}_{a'_k} > t \wedge \inf\{T^{\xi'}_{a_i}, T^{\xi'}_{a_m}\} > t)
\end{aligned}$$

$$= P'^{\bullet}(\bigwedge_{k=1,\ldots,n+1} T^{\xi'}_{a_k} \leq x_k + t \mid \bigwedge_{k=1,\ldots,n+1} T^{\xi'}_{a_k} > t).$$

2nd case – $x_{n+1} \leq \max\{x_i, x_m\}$:

$$\begin{aligned}
P'^{\bullet}(\bigwedge_{k=1,\ldots,n} & {T'}^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_k \wedge \inf\{{T'}^{\delta\xi'(\omega')(t)}_{a'_i}, {T'}^{\delta\xi'(\omega')(t)}_{a'_m}\} \leq x_{n+1}) \\
&= P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq i}} {T'}^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_k \wedge {T'}^{\delta\xi'(\omega')(t)}_{a'_i} \leq x_{n+1}) \\
&\quad + P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq m}} {T'}^{\delta\xi'(\omega')(t)}_{a'_k} \wedge {T'}^{\delta\xi'(\omega')(t)}_{a'_m} \leq x_{n+1}) \\
&\quad - P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq i,m}} {T'}^{\delta\xi'(\omega')(t)}_{a'_k} \wedge \bigwedge_{k=i,m} {T'}^{\delta\xi'(\omega')(t)}_{a'_k} \leq x_{n+1}) \\
&= P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq i}} {T'}^{\xi'}_{a'_k} \leq x_k + t \wedge {T'}^{\xi'}_{a'_i} \leq x_{n+1} + t \mid \bigwedge_{k=1,\ldots,n} {T'}^{\xi'}_{a'_k} > t) \\
&\quad + P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq m}} {T'}^{\xi'}_{a'_k} \leq x_k + t \wedge {T'}^{\xi'}_{a'_m} \leq x_{n+1} + t \mid \bigwedge_{k=1,\ldots,n} {T'}^{\xi'}_{a'_k} > t) \\
&\quad - P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq i,m}} {T'}^{\xi'}_{a'_k} \leq x_k + t \wedge \bigwedge_{k=i,m} {T'}^{\xi'}_{a'_k} \leq x_{n+1} + t \mid \bigwedge_{k=1,\ldots,n} {T'}^{\xi'}_{a'_k} > t) \\
&= P'^{\bullet}(\bigwedge_{\substack{k=1,\ldots,n \\ k\neq i,m}} {T'}^{\xi'}_{a'_k} \leq x_k + t \wedge \inf\{{T'}^{\xi'}_{a'_i}, {T'}^{\xi'}_{a'_m}\} \leq x_{n+1} + t \\
&\qquad \mid \bigwedge_{k=1,\ldots,n} {T'}^{\xi'}_{a'_k} > t \wedge \inf\{{T'}^{\xi'}_{a'_i}, {T'}^{\xi'}_{a'_m}\} > t) \\
&= P'^{\bullet}(\bigwedge_{k=1,\ldots,n+1} T^{\xi'}_{a_k} \leq x_k + t \mid \bigwedge_{k=1,\ldots,n+1} T^{\xi'}_{a_k} > t). \qquad \square
\end{aligned}$$

4.4 Calling

We are now ready to describe calling as a quite useful form of interconnecting rta's. Assume we want to combine two given rta's m' and m'' while imposing the following interaction between them: some action b' from m' 'causes' some action b'' from m''. We first calculate the product $m' \otimes m''$ and then obtain the envisaged rta by cartesian lifting of the action morphism from the resulting alphabet $A' \uplus A''$ to $A' \uplus A''$ such that $h(a) = \{a\}$ for $a \neq b''$ and $h(b'') = \{b', b''\}$. That is, action b'' is 'caused' by b' besides itself.

Example 33 *Lightbulb and child.* As an illustration consider the example depicted in Figure 4 of interaction between the lightbulb described in Example 2 and a child (a very narrow-minded one that lives for breaking the bulb) where

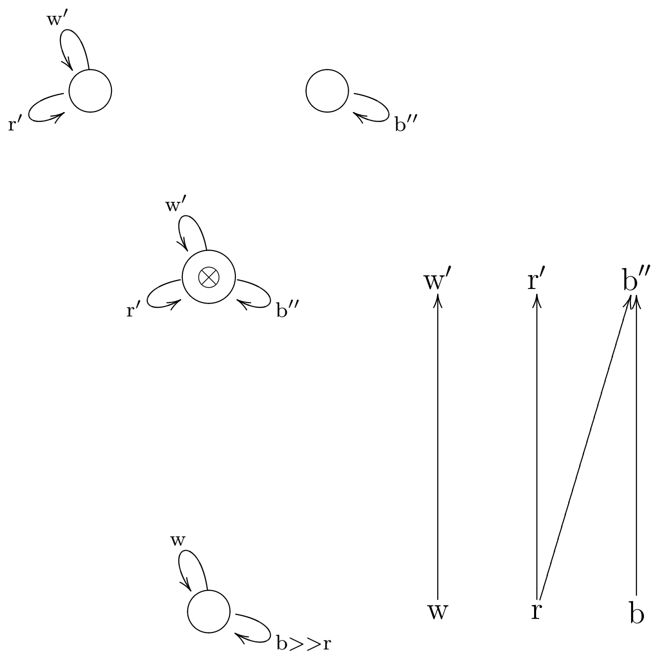

Fig. 4. Interconnection by action calling

b'' (break) 'causes' r' (replacement). Note that the hiding as a co-cartesian lifting presented in Subsection 4.2 would allow us to hide the break action from the resulting rta above, since in that rta the action replacement fulfills requirement 2 for the existence of the co-cartesian lifting. However, the same desideratum could be achieved when setting up the cartesian lifting above by dropping action b from the alphabet of actions. Clearly, hiding can be achieved either way, but always under very restrictive conditions.

Observe that for Example 33 both the lightbulb and the child can be modeled by an sta (provided that the distributions of breaking and resetting are exponential). However neither aggregation nor interconnection by action calling are sta's when we assume that breaking is not exponential. In aggregation, because we cannot record the cumulative amount of time since the child has broken the bulb for the last time. In interconnection, because the random variables corresponding to breaking and replacing are no longer independent.

5 Unfolding

We now turn our attention to the problem of obtaining the behaviour of an rta. For the sake of simplicity, we consider only DTC-rta's (that is, rta's which all randomization is on the times of the actions and not in the choice of configuration) however, the theory can be smoothly adapted to include all rta's by incorporating the results of [27], where the probabilistic automata considered have probabilistic choice but do not have random timed actions. As expected, given the random nature of randomly timed automata, the behaviour is a stochastic process. More precisely, it is a point process of a very specific kind that we proceed to define:

Definition 34 A *run process* is a tuple $\langle \Sigma, \mathcal{P}, A, V\rangle$ where:

- Σ is a countable set (of random sources);
- $\mathcal{P} = \{\mathcal{P}_\sigma\}_{\sigma\in\Sigma}$ where each $\mathcal{P}_\sigma = \langle \Omega_\sigma, \mathcal{F}_\sigma, P_\sigma\rangle$ is a probability space;
- A is a finite set (of actions);
- $V : (\mathbb{R}^+)^{\mathbb{N}^+} \times (\Omega^\bullet)^{\mathbb{N}^+} \to ((\mathbb{R}^+)^A)^{\mathbb{N}^+}$ is $(\mathcal{P}^\bullet)^{\mathbb{N}^+}$-measurable.

Given a sequence $\hat\tau$ of (external) cut times and a sequence $\hat\omega$ of outcomes, $V(\hat\tau, \hat\omega)$ gives the sequence of next action times. That is, $V(\hat\tau, \hat\omega)_{na}$ gives the time when action a will occur after the n-th cut point. Note that a process can be observed only a countable number of times, thus explaining the role of $\mathbb{N}^+$.

In more technical terms, V is a run process corresponding to a $(\mathbb{R}^+)^{\mathbb{N}^+}$-parameterized point process over the probability space $(\mathcal{P}^\bullet)^{\mathbb{N}^+}$, taking values in $(\mathbb{R}^+)^A$. More precisely, for each $\hat\tau \in (\mathbb{R}^+)^{\mathbb{N}^+}$, $\{V(\hat\tau, .)_n\}_{n\in\mathbb{N}^+}$ is a stochastic process on $(\mathcal{P}^\bullet)^{\mathbb{N}^+}$ taking values in $(\mathbb{R}^+)^A$. As we will see latter on, the parameterization is necessary to cope with aggregation and interconnection. From any run process we can derive the following processes:

- Step process:
  · $X : (\mathbb{R}^+)^{\mathbb{N}^+} \times (\Omega^\bullet)^{\mathbb{N}^+} \to (\mathbb{R}^+)^{\mathbb{N}^+}$;
  · $X(\hat\tau, \hat\omega)_n = \min\{\hat\tau_n, \inf_{a\in A} V(\hat\tau, \hat\omega)_{na}\}$.
  Given a sequence $\hat\tau$ of (external) cut times and a sequence $\hat\omega$ of outcomes, $X(\hat\tau, \hat\omega)$ gives the sequence of next cut times either external or internal. That is, $X(\hat\tau, \hat\omega)_n$ gives the minimum between the external cut time $\hat\tau_n$ and the time for the next occurrence of an action.
- Mark process:
  · $K : (\mathbb{R}^+)^{\mathbb{N}^+} \times (\Omega^\bullet)^{\mathbb{N}^+} \to (\wp A)^{\mathbb{N}^+}$;
  · $K(\hat\tau, \hat\omega)_n = \{a \in A : V(\hat\tau, \hat\omega)_{na} = X(\hat\tau, \hat\omega)_n\}$.
  Given a sequence $\hat\tau$ of (external) cut times and a sequence $\hat\omega$ of outcomes, $K(\hat\tau, \hat\omega)$ gives the sequence of sets of occurring actions. That is, $K(\hat\tau, \hat\omega)_n$ gives the set of actions occurring after the n-th cut point.
- Configuration process:
  · $Z : (\mathbb{R}^+)^{\mathbb{N}^+} \times (\Omega^\bullet)^{\mathbb{N}^+} \to ((\mathbb{R}^+ \times \wp A)^*)^{\mathbb{N}}$;
  · $Z(\hat\tau, \hat\omega)_0 = \epsilon$;
  · $Z(\hat\tau, \hat\omega)_{n+1} = Z(\hat\tau, \hat\omega)_n \langle X(\hat\tau, \hat\omega)_{n+1}, K(\hat\tau, \hat\omega)_{n+1}\rangle$.
  Given a sequence $\hat\tau$ of (external) cut times and a sequence $\hat\omega$ of outcomes, $Z(\hat\tau, \hat\omega)$ gives the sequence of finite sequences of pairs (step,mark). That is, $Z(\hat\tau, \hat\omega)_n$ gives the sequence of all pairs (step,mark) occurring until the n-th cut point (inclusive).

$$Z_0 \xrightarrow[\hat{\tau}_1\,\hat{\omega}_1]{V_1\,X_1} Z_1\,K_1 \xrightarrow[\hat{\tau}_2\,\hat{\omega}_2]{V_2\,X_2} Z_2\,K_2 \xrightarrow[\cdots]{\cdots}$$

Fig. 5. Derived processes

Clearly, this notion of run process is quite general. For instance, nothing prevents dependence on the future. But for the purpose of describing the behaviour of an rta it is enough to work with realizable run processes, defined as follows:

Definition 35 A *realizable* run process (rrp) is a run process such that

$$V(\hat{\tau},\hat{\omega})_n = V(\hat{\tau}',\hat{\omega}')_n$$

whenever $Z(\hat{\tau},\hat{\omega})_{n-1} = Z(\hat{\tau}',\hat{\omega}')_{n-1}$ and $\hat{\omega}_n = \hat{\omega}'_n$.

We now proceed to establish the envisaged adjunction between rta's and their behaviours. We have to obtain two functors: the folding functor going from behaviors to automata and the unfolding functor going from automata to behaviors. We start by looking at the folding map F.

Proposition 36 Given an rrp $\varphi = \langle \Sigma, \mathcal{P}, A, V\rangle$, the folding of φ,

$$F(\varphi) = \langle \Sigma, \mathcal{P}, A, \Xi, T, \delta\rangle$$

with:

- $\Xi = \{Z(\hat{\tau},\hat{\omega})_n : n \in \mathbb{N}, \hat{\tau} \in (\mathbb{R}^+)^{\mathbb{N}^+}, \hat{\omega} \in (\Omega^\bullet)^{\mathbb{N}^+}\}$;
- $\xi_0 = \epsilon$;
- $T^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1}) = V(\hat{\tau},\hat{\omega})_{n+1}$;
- $\delta^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1})(\tau_{n+1}) = Z(\hat{\tau},\hat{\omega})_{n+1}$,

is a DTC-rta.

PROOF. First, observe that δ is well defined since φ is a realizable run process. So, it is enough to show that $F(\varphi)$ is a DTC-rta. We start by checking that $F(\varphi)$ holds the DTBW condition. Suppose that we have $\tau_{n+1} < \min\{\mu^{Z(\hat{\tau},\hat{\omega}')_n}(\omega'_{n+1}), \mu^{Z(\hat{\tau},\hat{\omega}'')_n}(\omega''_{n+1})\}$ and without loss of generality that $\hat{\omega}'_{n]} = \hat{\omega}''_{n]}$. Then $Z(\hat{\tau},\hat{\omega}')_n = Z(\hat{\tau},\hat{\omega}'')_n$ and furthermore $X(\hat{\tau},\hat{\omega}')_{n+1} = X(\hat{\tau},\hat{\omega}'')_{n+1} = \tau_{n+1}$. Moreover $K(\hat{\tau},\hat{\omega}')_{n+1} = \emptyset = K(\hat{\tau},\hat{\omega}'')_{n+1}$ and thus

$$\begin{aligned}\delta^{Z(\hat{\tau},\hat{\omega}')_n}(\hat{\omega}'_{n+1})(\hat{\tau}_{n+1}) &= Z(\hat{\tau},\hat{\omega}')_{n+1} \\ &= Z(\hat{\tau},\hat{\omega}'')_{n+1} \\ &= \delta^{Z(\hat{\tau},\hat{\omega}'')_n}(\hat{\omega}''_{n+1})(\hat{\tau}_{n+1}).\end{aligned}$$

Hence $F(\varphi)$ holds the DTBW condition. Secondly, we show that $F(\varphi)$ holds the DTAW condition. Suppose that

$$\tau_{n+1} = \mu^{Z(\hat{\tau},\hat{\omega}')_n}(\omega'_{n+1}) = \mu^{Z(\hat{\tau},\hat{\omega}'')_n}(\omega''_{n+1})$$

and without loss of generality that $\hat{\omega}'_{n]} = \hat{\omega}''_{n]}$. Then $Z(\hat{\tau},\hat{\omega}')_n = Z(\hat{\tau},\hat{\omega}'')_n$ and furthermore $X(\hat{\tau},\hat{\omega}')_{n+1} = X(\hat{\tau},\hat{\omega}'')_{n+1} = \tau_{n+1}$. Moreover, if

$$\alpha^{Z(\hat{\tau},\hat{\omega}')_n}(\omega'_{n+1})(\mu^{Z(\hat{\tau},\hat{\omega}')_n}(\omega'_{n+1})) = \alpha^{Z(\hat{\tau},\hat{\omega}'')_n}(\omega''_{n+1})(\mu^{Z(\hat{\tau},\hat{\omega}'')_n}(\omega''_{n+1}))$$

then $K(\hat{\tau},\hat{\omega}')_{n+1} = K(\hat{\tau},\hat{\omega}'')_{n+1}$ and therefore

$$\begin{aligned}\delta^{Z(\hat{\tau},\hat{\omega}')_n}(\hat{\omega}'_{n+1})(\hat{\tau}_{n+1}) &= Z(\hat{\tau},\hat{\omega}')_{n+1}\\ &= Z(\hat{\tau},\hat{\omega}'')_{n+1}\\ &= \delta^{Z(\hat{\tau},\hat{\omega}'')_n}(\hat{\omega}''_{n+1})(\hat{\tau}_{n+1}).\end{aligned}$$

Thus $F(\varphi)$ is a DTC-rta. □

The set of configurations in the rta $F(\varphi)$ corresponds to the set of reachable states of the configuration process. Hence, the initial configuration is ϵ. The random winning times are the sequences of next action times. Finally, the random transition $\delta^{Z(\hat{\tau},\hat{\omega})_n}$ is the $(n+1)$-th projection of the configuration process.

The appropriate notion of rrp morphism is easily obtained by adapting the notion of rta morphism.

Definition 37 An *rrp morphism* $h : \varphi \to \varphi'$ is a pair $\langle h^s, h^a \rangle$ where:

- $h^s : \Sigma' \to \Sigma$ in **Set**;
- $h^a : A \to A'$ in $\mathbf{Rel}^-$;

such that:

(1) $\mathcal{P}'_{\sigma'} = \mathcal{P}_{h^s(\sigma')}$;
(2) for every $a' \in h^a(a)$, $V(\hat{\tau},\hat{\omega})_{na} \leq V'(X(\hat{\tau},\hat{\omega}),\hat{\omega} \circ h^s)_{na'}$.

Realizable run processes together with their morphisms constitute the category **rRun**. The following result gives us some information about how rrp morphisms appear at the levels of both the step process and the mark process.

Proposition 38 Let $h : \varphi \to \varphi'$ be a morphism in **rRun**. Then:

- $X(\hat{\tau},\hat{\omega})_n = X'(X(\hat{\tau},\hat{\omega}),\hat{\omega} \circ h^s)_n$;
- $h^a(K(\hat{\tau},\hat{\omega})_n) \supseteq K'(X(\hat{\tau},\hat{\omega}),\hat{\omega} \circ h^s)_n$.

PROOF.
(i) We first check that $X(\hat{\tau},\hat{\omega})_n = X'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n$:

$$\begin{aligned}
&X'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n\\
&\quad = X'(\min\{\hat{\tau}_n, \inf_{a\in A}\{V(\hat{\tau},\hat{\omega})_{na}\}\},\hat{\omega}\circ h^s)_n\\
&\quad = \min\{\hat{\tau}_n, \inf_{a\in A}\{V(\hat{\tau},\hat{\omega})_{na}\}, \inf_{a'\in A'}\{V'(\min\{\hat{\tau}_n, \inf_{a\in A}\{V(\hat{\tau},\hat{\omega})_{na}\}\},\hat{\omega}\circ h^s)_{na'}\}\}\\
&\quad = \min\{\hat{\tau}_n, \inf_{a\in A}\{V(\hat{\tau},\hat{\omega})_{na}\}\}\\
&\quad = X(\hat{\tau},\hat{\omega})_n.
\end{aligned}$$

(ii) We start by showing that

$$\begin{aligned}
&K'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n\\
&\qquad = \{a'\in A' : \exists_{a\in K(\hat{\tau},\hat{\omega})_n}\, a'\in h^s(a)\wedge V(\hat{\tau},\hat{\omega})_{na} = V'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'}\}.
\end{aligned}$$

Let $a'\in K'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n$. Since h^s is a map in $\mathbf{Rel}^-$ there exists $a\in A$ such that $a'\in h^s(a)$. Furthermore,

$$\begin{aligned}
V(\hat{\tau},\hat{\omega})_{na} &\leq V'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'}\\
&= X'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n\\
&= X(\hat{\tau},\hat{\omega})_n.
\end{aligned}$$

By definition of X, $V(\hat{\tau},\hat{\omega})_{na} = X(\hat{\tau},\hat{\omega})_n$ and hence

$$a'\in\{a'\in A' : \exists_{a\in K(\hat{\tau},\hat{\omega})_n}\, a'\in h^s(a)\wedge V(\hat{\tau},\hat{\omega})_{na} = V'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'}\}.$$

Let $a'\in\{a'\in A' : \exists_{a\in K(\hat{\tau},\hat{\omega})_n}\, a'\in h^s(a)\wedge V(\hat{\tau},\hat{\omega})_{na} = V(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'}\}$. Then there is an $a\in K(\hat{\tau},\hat{\omega})_n$ such that $a'\in h^s(a)$. Furthermore

$$\begin{aligned}
V'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'} &= V(\hat{\tau},\hat{\omega})_{na}\\
&= X(\hat{\tau},\hat{\omega})_n\\
&= X'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n.
\end{aligned}$$

Therefore, $a'\in K'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n$. It is straightforward to conclude that

$$\begin{aligned}
&h^s(K(\hat{\tau},\hat{\omega})_n)\\
&\qquad \supseteq \{a'\in A' : \exists_{a\in K(\hat{\tau},\hat{\omega})_n}\, a'\in h^s(a)\wedge V(\hat{\tau},\hat{\omega})_{na} = V'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_{na'}\}.
\end{aligned}$$

□

Before turning our attention to unfolding, we conclude the analysis of folding by establishing the envisaged folding functor.

Proposition 39 The folding map F extends to a functor from **rRun** to **dRta** as follows:

- $F(\langle h^s, h^a \rangle) = \langle h^s, h^a, h^c \rangle$

where

- $h^c(Z(\hat{\tau}, \hat{\omega})_n) = Z'(X(\hat{\tau}, \hat{\omega}), \hat{\omega} \circ h^s)_n$.

PROOF. Let $\langle h^s, h^a \rangle : \varphi \to \varphi'$, we check that $F(\langle h^s, h^a \rangle)$ is a morphism in **dRta** from $F(\varphi)$ to $F(\varphi')$. Let $a' \in h^a(a)$. Then, for $n \geq 0$:

$$\begin{aligned} T_a^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1}) &= V(\hat{\tau}, \hat{\omega})_{n+1a} \\ &\leq V'(X(\hat{\tau}, \hat{\omega}), \hat{\omega} \circ h^s)_{n+1a'} \\ &= T_{a'}^{Z'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n}(\hat{\omega}_{n+1} \circ h^s) \\ &= T_{a'}^{h^c(Z(\hat{\tau},\hat{\omega}))_n}(\hat{\omega}_{n+1} \circ h^s). \end{aligned}$$

Furthermore we have:

$$\begin{aligned} h^c(\delta^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1})(X(\hat{\tau}, \hat{\omega})_{n+1})) &= h^c(Z(\hat{\tau}, \hat{\omega})_{n+1}) \\ &= Z'(X(\hat{\tau}, \hat{\omega}), \hat{\omega} \circ h^s)_{n+1} \\ &= \delta'^{Z'(X(\hat{\tau},\hat{\omega}),\hat{\omega}\circ h^s)_n}(\hat{\omega}_{n+1} \circ h^s)(X(\hat{\tau}, \hat{\omega})_{n+1}) \\ &= \delta'^{h^c(Z(\hat{\tau},\hat{\omega})_n)}(\hat{\omega}_{n+1} \circ h^s)(X(\hat{\tau}, \hat{\omega})_{n+1}). \qquad \square \end{aligned}$$

It is useful to denote by $\hat{\omega}_{n]}$ the sequence of the n first elements of $\hat{\omega} \in (\Omega^\bullet)^{\mathbb{N}^+}$. Then, the *unfolding functor* G is easily established:

Proposition 40 The maps assigning

- to each DTC-rta $m = \langle \Sigma, \mathcal{P}, A, \Xi, T, \delta \rangle$ the rrp $G(m) = \langle \Sigma, \mathcal{P}, A, V \rangle$ with:
  · $V(\hat{\tau}, \hat{\omega})_n = T^{\delta^{*\xi_0}(\hat{\omega}_{n-1]})(X(\hat{\tau},\hat{\omega})_{n-1]})}(\hat{\omega}_n)$;
- and to each rta morphism $\langle h^s, h^a, h^c \rangle$ the rrp morphism $\langle h^s, h^a \rangle$;

constitute a functor $G : \mathbf{dRta} \to \mathbf{rRun}$ (the unfolding functor).

PROOF. Let m be a DTC-rta. We show that $G(m)$ is a rrp. We start by showing an auxiliary result:

If $Z(\hat{\tau}, \hat{\omega})_n = Z(\hat{\tau}', \hat{\omega}')_n$ then $\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau}, \hat{\omega})_{n]}) = \delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}', \hat{\omega}')_{n]})$.

We show this result by induction on n.
(i) Base: $\delta^{*\xi_0}(\hat{\omega}_{0]})(\epsilon) = \xi_0 = \delta^{*\xi_0}(\hat{\omega}'_{0]})(\epsilon)$.
(ii) Induction step: Suppose that $Z(\hat{\tau}, \hat{\omega})_{n+1} = Z(\hat{\tau}', \hat{\omega}')_{n+1}$, then clearly $Z(\hat{\tau}, \hat{\omega})_n = Z(\hat{\tau}', \hat{\omega}')_n$, and so

$$\delta^{*\xi_0}(\hat{\omega}_{n+1]})(X(\hat{\tau}, \hat{\omega})_{n+1]}) = \delta^{\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau},\hat{\omega})_{n]})}(\hat{\omega}_{n+1})(X(\hat{\tau}, \hat{\omega})_{n+1})$$

$$= \delta^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1})(X(\hat{\tau},\hat{\omega})_{n+1}).$$

Two cases must be considered:
a)

$$X(\hat{\tau},\hat{\omega})_{n+1} < \mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1}).$$

Observe that since $K(\hat{\tau},\hat{\omega})_{n+1} = \emptyset = K(\hat{\tau}',\hat{\omega}')_{n+1}$ we also have $X(\hat{\tau},\hat{\omega})_{n+1} < \mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}'_{n+1})$ and so, since m verifies the DTBW condition,

$$\begin{aligned}\delta^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1})(X(\hat{\tau},\hat{\omega})_{n+1}) &= \delta^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}'_{n+1})(X(\hat{\tau}',\hat{\omega}')_{n+1}) \\ &= \delta^{*\xi_0}(\hat{\omega}'_{n+1]})(X(\hat{\tau}',\hat{\omega}')_{n+1]}).\end{aligned}$$

b)

$$X(\hat{\tau},\hat{\omega})_{n+1} = \mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1}).$$

Note that since $K(\hat{\tau},\hat{\omega})_{n+1} = K(\hat{\tau}',\hat{\omega}')_{n+1}$ we also have

$$X(\hat{\tau},\hat{\omega})_{n+1} = \mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}'_{n+1})$$

and furthermore

$$\begin{aligned}K(\hat{\tau},\hat{\omega})_{n+1} &= \alpha^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1})(\mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})})(\hat{\omega}_{n+1})) \\ &= \alpha^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}'_{n+1})(\mu^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})})(\hat{\omega}'_{n+1})).\end{aligned}$$

Since m verifies the DTAW condition

$$\begin{aligned}\delta^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}_{n+1})(X(\hat{\tau},\hat{\omega})_{n+1}) &= \delta^{\delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}',\hat{\omega}')_{n]})}(\hat{\omega}'_{n+1})(X(\hat{\tau}',\hat{\omega}')_{n+1}) \\ &= \delta^{*\xi_0}(\hat{\omega}'_{n+1]})(X(\hat{\tau}',\hat{\omega}')_{n+1]}).\end{aligned}$$

We are now able to show that $G(m)$ is a rrp. Suppose that $Z(\hat{\tau},\hat{\omega})_{n-1} = Z(\hat{\tau}',\hat{\omega}')_{n-1}$ and $\hat{\omega}_n = \hat{\omega}'_n$. Then

$$\begin{aligned}V(\hat{\tau},\hat{\omega})_n &= T^{\delta^{*\xi_0}(\hat{\omega}_{n-1]})(X(\hat{\tau},\hat{\omega})_{n-1]})}(\hat{\omega}_n) \\ &= T^{\delta^{*\xi_0}(\hat{\omega}'_{n-1]})(X(\hat{\tau}',\hat{\omega}')_{n-1]})}(\hat{\omega}'_n) \\ &= V(\hat{\tau}',\hat{\omega}')_n. \qquad \square\end{aligned}$$

The n-th projection of the process of sequences of next action times V of $G(m)$ is easily obtained from the winning times in configuration $\delta^{*\xi_0}(\hat{\omega}_{n-1]})$.

Finally, we prove the main result of this section, extending to rta's the classical result on realization:

Theorem 41 The functor F is left adjoint to G.

PROOF.

(i) We start by giving a candidate for the co-unit. Let $m = \langle \Sigma, \mathcal{P}, A, \Xi, T, \delta \rangle$ be a DTC-rta. Consider $\varepsilon_m : F \circ G(m) \to m$ such that ${\varepsilon^s}_m = \mathrm{id}_\Sigma$, ${\varepsilon^a}_m = \mathrm{id}_A$ and ${\varepsilon^c}_m(Z(\hat{\tau}, \hat{\omega})_n) = \delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau}, \hat{\omega})_{n]})$. First, we check that ${\varepsilon^c}_m$ is well defined. Let $Z(\hat{\tau}, \hat{\omega})_n = Z(\hat{\tau}', \hat{\omega}')_{n'}$. Then, by construction of Z, $n = n'$ and $Z(\hat{\tau}, \hat{\omega})_i = Z(\hat{\tau}', \hat{\omega}')_i$ for any $i \in \{0 \dots n\}$. Furthermore, since $G(m)$ is realizable, we have that $V(\hat{\tau}, \hat{\omega})_{i+1} = V(\hat{\tau}', \hat{\omega}')_{i+1}$ for any $i \in \{0 \dots n\}$. Thus, by definition of V in $G(m)$, we conclude that $\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau}, \hat{\omega})_{n]}) = \delta^{*\xi_0}(\hat{\omega}'_{n]})(X(\hat{\tau}', \hat{\omega}')_{n]})$. So, $\varepsilon^c(Z(\hat{\tau}, \hat{\omega})_n) = \varepsilon^c(Z(\hat{\tau}', \hat{\omega}')_{n'})$. We continue by showing that ε_m is a morphism:

$$\begin{aligned} T_a^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1}) &= V(\hat{\tau}, \hat{\omega})_{n+1a} \\ &= V(X(\hat{\tau}, \hat{\omega}), \hat{\omega})_{n+1a} \\ &= T^{\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau},\hat{\omega})_{n]})}(\hat{\omega}_{n+1}) \\ &= T_a^{\varepsilon^c(Z(\hat{\tau},\hat{\omega})_n)}(\hat{\omega}_{n+1}); \end{aligned}$$

and

$$\begin{aligned} {\varepsilon^c}_m(\delta^{Z(\hat{\tau},\hat{\omega})_n}(\hat{\omega}_{n+1})(X(\hat{\omega}, \hat{\tau})_{n+1})) &= \delta^{*\xi_0}(\hat{\omega}_{n+1]})(X(\hat{\omega}, \hat{\tau})_{n+1]}) \\ &= \delta^{\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\omega},\hat{\tau})_{n]})}(\hat{\omega}_{n+1})(X(\hat{\omega}, \hat{\tau})_{n+1}) \\ &= \delta^{{\varepsilon^c}_m(Z(\hat{\tau},\hat{\omega})_n)}(\hat{\omega}_{n+1})(X(\hat{\omega}, \hat{\tau})_{n+1}). \end{aligned}$$

Furthermore, we check that ε_m is a natural transformation. Let $h : m \to m'$; then we only show that

$$\begin{aligned} h^c({\varepsilon^c}_m(Z(\hat{\tau}, \hat{\omega})_n)) &= h^c(\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\omega}, \hat{\tau})_{n]})) \\ &= \delta'^{*\xi'_0}(\hat{\omega}_{n]} \circ h^s)(X(\hat{\omega}, \hat{\tau})_{n]}) \\ &= {\varepsilon^c}_{m'}(Z'(X(\hat{\tau}, \hat{\omega}), \hat{\omega} \circ h^s)_n) \\ &= {\varepsilon^c}_{m'} \circ (F \circ G)(Z(\hat{\tau}, \hat{\omega})_n)). \end{aligned}$$

(ii) We finally check the universal property. Let $h : F(\varphi) \to m'$ be a morphism in **dRta**; then take $h' : \varphi \to G(m')$ where $h' = \langle h^s, h^a \rangle$. We first show that h' is morphism in **rRun**. Let $a' \in h^a(a)$; then

$$\begin{aligned} V(\hat{\tau}, \hat{\omega})_{na} &= T_a^{Z(\hat{\tau},\hat{\omega})_{n-1}}(\hat{\omega}_n) \\ &\leq {T'}_{a'}^{h^c(Z(\hat{\tau},\hat{\omega})_{n-1})}(\hat{\omega}_n \circ h^s) \\ &= {T'}_{a'}^{h^c(\delta^{*\xi_0}(\hat{\omega}_{n-1]})(X(\hat{\tau},\hat{\omega})_{n-1]}))}(\hat{\omega}_n \circ h^s) \\ &= {T'}_{a'}^{\delta^{*\xi'_0}(\hat{\omega}_{n-1]} \circ h^s)(X(\hat{\tau},\hat{\omega})_{n-1]})}(\hat{\omega}_n \circ h^s) \\ &= V'(X(\hat{\tau}, \hat{\omega}), \hat{\omega} \circ h^s)_{na'}. \end{aligned}$$

It remains to show that $h = \varepsilon_{m'} \circ F(h')$:

$$\begin{aligned} h^c(Z(\hat{\tau},\hat{\omega})_n) &= h^c(\delta^{*\xi_0}(\hat{\omega}_{n]})(X(\hat{\tau},\hat{\omega})_{n]})) \\ &= \delta'^{*\xi'_0}(\hat{\omega}_{n]} \circ h^s)(X(\hat{\tau},\hat{\omega})_{n]})) \\ &= \varepsilon^c{}_{m'}(Z'(X(\hat{\tau},\hat{\omega}),\hat{\omega} \circ h^s)_n) \\ &= (\varepsilon^c{}_{m'} \circ F(h)^c)(Z(\hat{\tau},\hat{\omega})_n). \end{aligned}$$

The uniqueness requirement is straightforward. □

We conclude this section by using the theorem above for relating the universal constructions in the category of rta's with those in the category of their behaviours.

Corollary 42 The functor G preserves products of rta's.

That is, the behaviour of an aggregation (product) of rta's is the products of their behaviours. Before looking at the behaviour of interconnections, we need to introduce the forgetful functor *Mrk* from **rRun** to $\mathbf{Rel}^-$ that extracts the alphabets of actions and their maps from rrp's and their morphisms.

Theorem 43 The adjunction $F \dashv G$ is fibred with respect to *Mrk* and *Act*.

PROOF. Recall that an adjunction $L \dashv R$ is said to be *fibred* with respect to functors P and Q on the same base category iff: $Q \circ L = P$; $P \circ R = Q$; and $Q(\epsilon_X) = \mathrm{id}_{Q(X)}$. Note that the co-unit provided in Theorem 5.10, is vertical with respect to *Act*, that is, $\varepsilon^a{}_m = \mathrm{id}_A$. □

Corollary 44 The unfolding functor G preserves Cartesian liftings.

Therefore, the behaviour of the interconnection of two rta's is also obtained as the cartesian lifting on their behaviours.

6 Concluding remarks

The main goal of the paper – a categorical foundation of the theory of randomly timed automata – has been achieved: starting with the practical motivation and presentation of the envisaged notion, the categorical theory of minimization, aggregation, interconnection and realization of such automata has been developed. A very strict form of encapsulation has been also presented, albeit of little practical use, but helpful for technical reasons when establishing properties of interconnection. All these constructions have been presented universally: minimization and realization as adjunctions, aggregation as product, interconnection as cartesian lifting, and encapsulation of actions as co-cartesian lifting. With respect to interconnection, the special form known as 'action calling' deserved detailed analysis.

All these results show that part of the theory of classical automata extends to randomly timed automata. Given the practical interest of such automata for the purpose of modeling computer systems [9], the theory of randomly timed automata as developed in this paper has some significance from the point of view of Computer Science. Furthermore, from the point of view of Stochastic Processes, the ability to 'present' stochastic processes as the behaviour of randomly timed automata is expected to be of great interest: for instance, the interplay between two point processes that may be recognized as the behaviour of two such automata can be much better understood at the level of the machines. Along these lines, the categorical description of such a kind of interplay between point processes is an immediate contribution that may lead to further developments.

Another contribution is the presentation of stochastic timed automata by means of random sources and random variables, which led to the establishment of an embedding into the category of randomly timed automata. Such embedding provides a natural way to unfold and combine stochastic timed automata. Given that, with stochastic timed automata, it is not possible to introduce dependencies in the random times of actions, the interconnection of stochastic timed automata is not, in general, a stochastic timed automaton. Thus, the notion of randomly timed automaton is well justified as an extension of the notion of stochastic timed automaton where the combination theory is self-contained.

The probabilistic aspects of stochastic Petri nets [17,24,8] are similar to those of randomly timed automata: both have random times for transitions. Therefore, it will be interesting to establish a relationship between these automata and stochastic Petri nets, hoping to achieve for these probabilistic systems results similar to those relating classical automata (transition systems) to classical Petri nets (along the lines of [35]).

Another interesting line of research should be aimed at studying in detail special classes of randomly timed automata, namely those having action times without memory (that is, with exponential distribution). This class of memoryless random times is of great practical significance and should be more amenable to an effective stochastic analysis. To this end, the use of the embedded general state Markov process is the most promising line of development. In particular, we intend to apply techniques to rta's borrowed from Markov additive processes [29].

Finally, it is worthwhile to develop the categorical theory of rta's over the bi-categorical approach of Walters et al [21,22] and pursue other directions in which a categorical theoretical perspective would be interesting, besides investigating traditional results in automata. Moreover, it would be worthwhile to study the properties of an algebra for putting together rta's.

Acknowledgements

The authors wish to express their gratitude to Reinhard German, Javier Pinto and Richard Serfozo for many helpful discussions on relevant topics. We would also like to acknowledge the anonymous referes for helpful suggestions in improving the paper.

References

[1] J. Adámek. Realization theory for automata in categories. *J. Pure Appl. Algebra*, 9(3):281–296, 1976.

[2] J. Adámek and V. Trnková. *Automata and Algebras in Categories*. Kluwer Academic Publishers, 1991.

[3] L. Alfaro. How to specify and verify the long-run average behavior of probabilistic systems. In *LICS 98: 13th Annual IEEE Symposium on Logic in Computer Science*, pages 454–465. IEEE Press, 1998.

[4] L. Alfaro. Computing minimum and maximum reachability times in probabilistic systems. In J. Baeten and S. Mauw, editors, *CONCUR'99: Concurrency theory*, volume 1664 of *Lecture Notes in Computer Science*, pages 66–81. Springer-Verlag, 1999.

[5] M. Arbib and E. Manes. Adjoint machines, state-behaviour machines and duality. *J. Pure Appl. Algebra*, 6(3):313–344, 1975.

[6] M. Arbib and E. Manes. Machines in a category. *J. Pure Appl. Algebra*, 19:9–20, 1980.

[7] C. Baier and M. Kwiatkowska. On the verification of qualitative properties of probabilistic processes under fairness constraints. *Information Processing Letters*, 66(1):71–79, 1998.

[8] A. Bobbio, A. Puliafito, M. Telek, and K. Trivedi. Recent developments in non-Markovian stochastic Petri nets. *Journal of Systems Circuits and Computers*, 8(1):119–158, 1998.

[9] P. Buchholz. Exact performance equivalence: an equivalence relation for stochastic automata. *Theoretical Computer Science*, 215(1-2):263–287, 1999.

[10] C. Cassandras and S. Lafortune. *Introduction to Discrete Event Systems*. Kluwer Academic Publishers, 1999.

[11] E. Çinlar. *Introduction to Stochastic Processes*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[12] D. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer-Verlag, 1988.

[13] F. Demichelis and W. Zielonka. Controlled timed automata. In D. Sangiorgi and R. Simone, editors, *CONCUR'98: Concurrency theory*, volume 1466 of *Lecture Notes in Computer Science*, pages 455–469. Springer-Verlag, 1998.

[14] J. Desharnais, A. Edalat, and P. Panangaden. A logic characterization of bisimulation for labeled Markov processes. In *LICS 98: 13th Annual IEEE Symposium on Logic in Computer Science*, pages 478–487. IEEE Press, 1998.

[15] H. Ehrig, K. Kiermeier, H.-J. Kreowski, and W. Kühnel. *Universal Theory of Automata - A Categorical Approach*. Teubner, 1974.

[16] R. German. *Performance Analysis of Communication Systems, Modeling with Non-Markovian Stochastic Petri Nets*. John Wiley and Sons, 2000.

[17] R. German, A. Moorsel, M. Qureshi, and W. Sanders. Expected impulse rewards in Markov regenerative stochastic Petri nets. In *Application and theory of Petri nets 1996*, volume 1091 of *Lecture Notes in Computer Science*, pages 172–191. Springer-Verlag, 1996.

[18] J. Goguen. Minimal realization of machines in closed categories. *Bull. Amer. Math. Soc.*, 78:777–783, 1972.

[19] J. Goguen. Realization is universal. *Mathematical Systems Theory*, 6:359–374, 1972.

[20] T. Henzinger, Z. Manna, and A. Pnueli. Temporal proof methodologies for timed transition systems. *Information and Computation*, 112(2):273–337, 1994.

[21] P. Katis, N. Sabadini, and R. F. C. Walters. Representing Place/Transition nets in Span(Graph). In M. Johnson, editor, *Proceedings of the Sixth AMAST Conference*, volume 1349 of *Lecture Notes in Computer Science*, pages 323–336. Springer-Verlag, 1997.

[22] P. Katis, N. Sabadini, and R. F. C. Walters. Span(Graph): A categorical algebra of transition systems. In M. Johnson, editor, *Proceedings of the Sixth AMAST Conference*, volume 1349 of *Lecture Notes in Computer Science*, pages 307–322. Springer-Verlag, 1997.

[23] O. Maler. A decomposition theorem for probabilistic transition systems. *Theoretical Computer Science*, 145(1-2):391–396, 1995.

[24] M. Marsan, G. Balbo, A. Bobbio, G. Chiola, G. Conte, and A. Cumani. The effect of excecution policies on the semantics of stochastic Petri nets. *IEEE Trans. Softw. Engin.*, 15:832–846, 1989.

[25] P. Mateus. *Interconnection of Probabilistic Systems.* PhD thesis, IST, Universidade Técnica de Lisboa, 2000. Supervised by A. Sernadas and C. Sernadas.

[26] P. Mateus, A. Sernadas, and C. Sernadas. Precategories for combining probabilistic automata. *Electronic Notes in Theoretical Computer Science*, 29, 1999.

[27] P. Mateus, A. Sernadas, and C. Sernadas. Realization of probabilistic automata: Categorical approach. In Didier Bert and Christine Choppy, editors, *Recent Trends in Algebraic Development Techniques - Selected Papers*, volume 1827 of *Lecture Notes in Computer Science*, pages 237–251. Springer-Verlag, 2000.

[28] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* Springer-Verlag, 1993.

[29] A. Pacheco and N. Prabhu. A Markovian storage model. *Annals of Applied Probability*, 6(1):76–91, 1996.

[30] S. Port. *Theoretical Probability for Applications.* John Wiley & Sons Inc., 1994.

[31] L. Schröder and P. Mateus. Universal aspects of probabilistic automata. *Mathematical Structures in Computer Science*, 12(4):481–512, 2002.

[32] A. Sernadas, C. Sernadas, and C. Caleiro. Denotational semantics of object specification. *Acta Informatica*, 35:729–773, 1998.

[33] E. Stark and S. Smolka. Compositional analysis of expected delays in networks of probabilistic I/O automata. In *LICS 98: 13th Annual IEEE Symposium on Logic in Computer Science*, pages 466–477. IEEE Press, 1998.

[34] D. Williams. *Diffusions, Markov Processes, & Martingales I: Foundations.* John Wiley, 1979.

[35] G. Winskel and M. Nielsen. Models of concurrency. In D. Gabbay S. Abramsky and T. Maibaum, editors, *Handbook of Logic in Computer Science 4*, pages 1–148. Oxford Science Publications, 1995.

[36] S.-H. Wu, s. Smolka, and E. Stark. Composition and behaviors of probabilistic I/O automata. *Theoretical Computer Science*, 176(1-2):1–38, 1997.