QUANTALES AND OBSERVATIONAL SEMANTICS

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We illustrate the idea that quantales can be regarded as algebras of experimental observations on physical systems, and we give a survey of some research in computer science where this idea has been used. We extend the mathematical framework hitherto available so that it can be applied to more general systems than before, in particular to quantum systems and systems whose behaviour is partially unobservable.

1. Introduction

The purpose of this paper is to illustrate the idea that quantales can be regarded as models of abstract "experimental observations" on computational or physical systems, and to survey some recent work that applies this idea in computer science.

In [Mulvey 1986], where the name "quantale" was introduced, it is suggested that certain quantales may be interpreted as a logic for quantum mechanics whose conjunction, not necessarily commutative, expresses the temporal order in which properties of a system are verified: the conjunction $a\&b$ should be read "a and then b". This is also regarded as a natural generalization of an analogous logical description of the spectrum of commutative C*-algebras that appears in [Banaschewski and Mulvey 2000], where the conjunction is commutative and the quantales are in fact just frames.

The idea that the "arrow of time" might be expressed by means of a non-commutative conjunction is also a motivation in [Yetter 1990], where the aim is to provide a quantale based semantics for non-commutative variants of the linear logic of [Girard 1987] (see also [Rosenthal 1990]). The connection between quantales and linear logic is at the basis of much work that relates quantales to computer science, and in particular to concurrency, but it should be pointed out that in the usual ways in which (commutative) linear logic is applied to concurrency the formulas can be thought of as *states* of a computation, and the (multiplicative) conjunction represents *parallel composition* of systems. The quantale semantics of linear logic interprets formulas in a quantale, and thus such quantales are far from being algebras of observations in the sense that we want to convey in the present paper. A typical example of this situation can be found in [Engberg and Winskel 1990; Engberg and Winskel 1993].

The interpretation of quantales that we have in mind has its roots in the mathematical theory of programming initiated by Scott and Strachey (see, e.g. [Stoy 1977]). The topological form of this theory allows us to think of open sets of a suitable topological space as being properties of programs or, as [Smyth 1983] puts it, "semi-decidable properties". This view is further abstracted in [Abramsky 1987; Abramsky 1991], where frames are algebras of "finitely observable properties", and in [Vickers 1989], where propositional geometric logic is vividly presented as being a logic of "finite observations" (see [Vickers 1989, Ch. 2]). A simple example can be the following. Let $l \in \mathbb{R}$ be the length of a rod (of an idealized non-quantum mechanical world). The assertion that l lies in the open interval $(p,q) \subseteq \mathbb{R}$, with $p < q$ rational numbers, can be verified experimentally by placing the rod on a perfect ruler with one end of the rod at the beginning of the ruler. If $l \in (p,q)$ (i.e., the assertion is true), then we should in principle be able to verify in finite time that the other end of the rod is strictly between p and q, even if that requires us to use a powerful magnifying lens. However, assume that $l = p$. Then the assertion is false, but no matter how close we look, or how powerful a microscope we use, it is not possible to conclude in finite time that $l \notin (p,q)$, for regardless of when we stop measuring we may believe that by getting still a little closer we might eventually see some distance between the end of the rod and p. Hence, the assertion can be affirmed by performing an observation that lasts finite time, but it cannot necessarily be refuted in finite time. Generalizing this example, we are led to viewing open sets as being those properties which can be affirmed by "finite means" if and only if they are true.

Hence, in this topological interpretation the "abstract finite observations" should form frames and their logic should be propositional geometric logic, but this does not account for those situations in physics and computer science where a system can be affected by the way in which it is observed. In [Vickers 1989, p. 188] it is briefly suggested that in such situations quantales should provide the right generalization of frames. This view also indicates that the logic of finite observations should be a generalized propositional geometric logic, and it is consistent with the original logical interpretation of quantales as proposed by Mulvey.

In [Abramsky and Vickers 1993] these ideas are extensively applied to the theory of concurrent systems ("process semantics"), more precisely as a way of classifying several notions of process equivalence that exist in the literature. The underlying idea of this work is that certain right quantale modules can be regarded as computational systems, and left quantale modules describe the "capabilities" of the systems. This program is extended in [Resende 2000] by studying strong bisimulation equivalence, which is conspicuously absent in [Abramsky and Vickers 1993]. Also, in [Resende 2000] the framework is reformulated by means of the notion of *tropological system*, which generalizes the topological systems of [Vickers 1989] and provides a good standpoint for comparing different systems by means of suitable morphisms. Further extensions of this work can be found in [Resende 1999a], which provides a logical presentation of tropological systems that is used as a basis for a specification logic of computational systems, and where some notions of implementation and composition of systems are studied; and in [Resende 1999b], which addresses non-interleaving process semantics.

The present paper has two aims. One is to survey the applications to process semantics just described, and the other is to present the mathematical framework in a form that may help bring out connections to quantum physics, for instance via [Amira, Coecke and Stubbe 1998], where quantales of "inductions" on physical systems are studied in the context of the Geneva School operational approach (see [Piron 1976] for an overview, or [Moore 1999] for a recent survey). With this in mind, in §3 we present a generalized version of tropological system that can be applied to quantum systems, as opposed to the systems in [Abramsky and Vickers 1993; Resende 2000], which are inherently classical. Furthermore we also add the possibility that systems may have hidden unobservable behaviour, a possibility that was previously unaccounted for.

The rest of the paper is organized as follows. In §2 we introduce basic definitions and results about sup-lattices, quantales and quantale modules. Then §3 presents tropological systems. §4 presents the logic of [Resende 1999a] and establishes the connection to classical tropological systems, according to which these are essentially the models of the logic. §4 also includes an example of how systems can be described with the logic, in particular when time and space can be observed. Finally, §5 surveys the applications to process semantics. It necessarily contains few technical details, being essentially meant to give an overview of some basic ideas in concurrency and an introduction to the results in [Abramsky and Vickers 1993; Resende 2000; Resende 1999b].

2. Sup-lattices, quantales and modules

This section briefly introduces some technical concepts that will be used throughout the rest of the paper. For general references on the category theory and lattice theory we recommend respectively [Mac Lane 1971; Borceux and Stubbe 2000] and [Birkhoff 1967]. For sup-lattices see [Joyal and Tierney 1984]. For general references about frames and locales see [Johnstone 1982; Vickers 1989], and for quantales see [Rosenthal 1990; Paseka and Rosický 2000].

2.1. SUP-LATTICES

Recall that a lattice is complete if and only if it has all joins, but maps between complete lattices may preserve all joins without preserving meets. The category whose objects are the complete lattices and whose morphisms are the join preserving maps is denoted by **SL**. When a complete lattice is thought of as an object of **SL** it is called a *sup-lattice*, and a map that preserves all the joins is a *sup-lattice homomorphism*. The bottom element of a sup-lattice will be denoted by 0 and the top by $\top$. Any sup-lattice homomorphism preserves 0. A homomorphism that also preserves $\top$ is called *strong*. When necessary we use subscripts, e.g. writing $\top_L$ for the top of L, etc.

Example 2.1 Let X be a set. The powerset 2^X under the inclusion order is a sup-lattice, and it is freely generated by X.

Let L be a sup-lattice, and $R \subseteq L \times L$ a binary relation on L. We say that a map $f : L \to M$ *respects* R if $f(x) = f(y)$ for all $(x, y) \in R$. There exists a surjective sup-lattice homomorphism $j : L \to L'$ that respects R and such that any other sup-lattice homomorphism $h : L \to L''$ that respects R factors through j. In other words, L' is the *quotient* of L by the sup-lattice congruence generated by R. Given sets G and $R \subseteq 2^G \times 2^G$, the quotient of 2^G by the sup-lattice congruence generated by R is said to be *presented* by G and R and we denote it by $SL\langle G \mid R\rangle$; the elements of G are the *generators*, and the pairs $(X, Y) \in R$ are the *defining relations* of the presentation. Every sup-lattice L is isomorphic to a sup-lattice presented in this way.

If L is a sup-lattice we call L with the order reversed the *dual* of L and denote it by $\widehat{L}$. If $f : L \to M$ is a homomorphism of sup-lattices it has a right adjoint $f_* : M \to L$ that preserves all meets, and thus it is also a sup-lattice homomorphism $\widehat{f} : \widehat{M} \to \widehat{L}$, called the *dual* of f.

If L and M are sup-lattices, the set of sup-lattice homomorphisms $\mathbf{SL}(L, M)$ is itself a sup-lattice with order and joins defined pointwise, and it is order isomorphic to $\mathbf{SL}(\widehat{M}, \widehat{L})$. There is also an obvious order isomorphism $\widehat{L} \cong \mathbf{SL}(2, \widehat{L})$, where 2 denotes a sup-lattice with two elements. Hence, by duality we conclude that $\widehat{L} \cong \mathbf{SL}(L, 2)$ because $\widehat{2} \cong 2$. Explicitly, each $x \in \widehat{L}$ is assigned to the homomorphism $a_x : L \to 2$ (the "annihilator" of x), defined by

$$a_x(y) = \begin{cases} \top & \text{if } y \not\leq x \text{ (in } L) \\ 0 & \text{otherwise} \end{cases}$$

In practice we will often identify $\widehat{M}$ with $\mathbf{SL}(M, 2)$, according to convenience.

Let L, M and N be sup-lattices. A sup-lattice *bimorphism* $f : L \times M \to N$ is a map that preserves joins in each variable separately. There is a universal bimorphism $L \times M \to L \otimes M$, whose image $L \otimes M$ is the *tensor product* of L and M ($L \otimes M$ can be presented as $SL\langle L \times M \mid R\rangle$ for a suitable set R). This tensor product makes **SL** a symmetric monoidal closed category. The right adjoint to the tensor product is the hom-functor, and the natural isomorphism

$$\mathbf{SL}(L \otimes M, N) \cong \mathbf{SL}(L, \mathbf{SL}(M, N))$$

is an isomorphism of sup-lattices.

2.2. QUANTALES

A *quantale* Q is a sup-lattice equipped with an associative binary operation $(a, b) \mapsto a \cdot b$ that distributes over joins in both variables:

$$a \cdot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \cdot b_i) \qquad (\bigvee_{i \in I} a_i) \cdot b = \bigvee_{i \in I} (a_i \cdot b) \ .$$

The associative binary operation is called *multiplication*. If the multiplication has a unit the quantale is *unital*. The unit of a unital quantale is be denoted by 1.[1] In other words, a quantale (resp. unital quantale) is a semigroup (resp. monoid) in **SL**.

Let Q and Q' be quantales. A *homomorphism* $h : Q \to Q'$ is a map that is both a homomorphism of sup-lattices and semigroups. A homomorphism is *unital* if it preserves the unit and *pre-unital* if it satisfies $h(1_Q) \geq 1_{Q'}$ (such homomorphisms are called unital in [Mulvey and Pelletier 2000]). Henceforth we write **Qu** for the category of unital quantales and pre-unital homomorphisms, and $\mathbf{Qu}_1$ for the category of unital quantales with unital homomorphisms.

Example 2.2 1. Let Q be a quantale. Keeping the same order but defining a new multiplication $a \bullet b = b \cdot a$ we obtain another quantale, obviously unital if Q is, which we denote by Q^*.

2. Let M be a monoid. The powerset 2^M under the inclusion order and multiplication computed pointwise is a unital quantale, and it is freely generated by M as a monoid (i.e., any monoid homomorphism $f : M \to Q$ extends uniquely to a unital quantale homomorphism $f^\sharp : 2^M \to Q$).

3. Let A be a set. Then the set of languages over A, 2^{A^*}, is a unital quantale and it is freely generated by A.

4. Let L be a sup-lattice. The sup-lattice of all sup-lattice endomorphisms on L, $\mathbf{SL}(L, L)$, is a unital quantale whose multiplication is composition of functions and whose unit is the identity map. We denote this quantale by $\mathcal{Q}(L)$. In this paper we adopt the convention that multiplication is given by forward composition, i.e., $f \cdot g = f; g$, instead of $f \cdot g = f \circ g$.

5. Let X be a set. The set $2^{X \times X}$ of binary relations on X under the inclusion order is a unital quantale whose multiplication is composition of relations (again assumed forward) and whose unit is the identity relation on X. This quantale is isomorphic to $\mathcal{Q}(2^X)$. The isomorphism from $\mathcal{Q}(2^X)$ to $2^{X \times X}$ maps each endomorphism f to the relation $R_f = \{(x, y) \mid y \in f(\{x\})\}$.

Similarly to sup-lattices, quantales and unital quantales can be presented by generators and relations. We shall be interested only in the case of unital quantales. Given a unital quantale Q and a subset $R \subseteq Q \times Q$, there is a surjective

[1] This notation differs from the one in [Paseka and Rosický 2000], where 1 denotes the top and e denotes the unit of a unital quantale. The choice essentially depends on whether in the generalization of frames as quantales one decides to see the top of a frame become the top of a quantale or its unit. In the present paper we adopt the latter option.

unital homomorphism $j : Q \to Q'$ that respects R and such that any other unital homomorphism $h : Q \to Q''$ that respects R factors through j, i.e., Q' is the quotient of Q by the quantale congruence generated by R. When Q is free, i.e., $Q \cong 2^{G^*}$ for some set of generators G, Q' is said to be *presented* by G and R, and we denote it by $Qu_1\langle G \mid R\rangle$.

A *representation* of a quantale Q on a sup-lattice L is a quantale homomorphism $r : Q \to \mathcal{Q}(L)$. A *relational representation* of Q on a set X is a quantale homomorphism $r : Q \to 2^{X\times X}$.

2.3. QUANTALE MODULES

Let Q be a quantale. A *right module* M over Q, or simply a *right Q-module*, is a sup-lattice with a binary operation $\_\cdot\_ : M \times Q \to M$, satisfying

$$\begin{aligned} (\bigvee X)\cdot a &= \bigvee\{x\cdot a \mid x \in X\}\ , \\ x\cdot(\bigvee S) &= \bigvee\{x\cdot a \mid a \in S\}\ , \\ x\cdot(a\cdot b) &= (x\cdot a)\cdot b\ . \end{aligned}$$

The binary operation of the module is called the *action*. Notice that we use the same symbol, "·", for the multiplication of a quantale and for its action on a module. It is always easy to determine from the context which is the case at hand.

A right Q-module homomorphism $k : M \to M'$ is a sup-lattice homomorphism that also preserves the action, i.e., such that, for all $x \in M$ and $a \in Q$,

$$k(x\cdot a) = k(x)\cdot a\ .$$

Left Q-modules are defined analogously, but with an action $\_\cdot\_ : Q \times M \to M$.

A right Q-module M can be identified with a representation $r : Q \to \mathcal{Q}(M)$ defined by $r(a)(x) = x\cdot a$, and a left Q-module M can be identified with a representation $r : Q \to \mathcal{Q}(M)^*$. Accordingly, a module is said to be *strong*, *unital* or *pre-unital* according to whether the corresponding representation is a strong, unital or pre-unital homomorphism. For instance, a right module is pre-unital if and only if it satisfies $x\cdot 1 \geq x$.

Example 2.3 1. Any (unital) quantale is both a right and a left (unital) module over itself, with action given by multiplication on the right and on the left, respectively.

2. Let Q be a quantale and $a \in Q$. The set $Q\cdot a = \{b\cdot a \mid b \in Q\}$ is obviously a left Q-module, unital if Q is unital.

3. Let Q be a quantale. Recall that an element $a \in Q$ is *right-sided* if $a\cdot\top \leq a$. If a is right-sided then $b\cdot a$ is right-sided for all $b \in Q$, and thus the set $\mathcal{R}(Q)$ of right-sided elements of Q is a left Q-module. If Q is unital then $\mathcal{R}(Q) = Q\cdot\top$.

4. Let Q be a unital quantale. Then $\mathcal{R}(Q)$ is initial in the subcategory of unital left Q-modules with strong homomorphisms [Resende 2000], and the same holds for pre-unital modules.

5. Any sup-lattice L is a unital right module over $\mathcal{Q}(L)$ and a unital left module over $\mathcal{Q}(L)^*$.

Let $h : Q \to Q'$ be a quantale homomorphism. Then any representation $r : Q' \to \mathcal{Q}(L)$ determines a representation $r \circ h$ of Q on L. Hence, any right Q'-module is also a right Q-module with action given by $x \bullet a = x \cdot h(a)$. If Q and Q' are unital and h is pre-unital (resp. unital) then any pre-unital (resp. unital) Q'-module becomes a pre-unital (resp. unital) Q-module. Similar remarks apply to left modules.

Let M be a right module over a quantale Q. Then the dual $\widehat{M}$ is a left Q-module, for $\mathcal{Q}(M) \cong \mathcal{Q}(\widehat{M})^*$. Explicitly, we have $(a \cdot f)(x) = f(x \cdot a)$ for all $a \in Q$, $f \in \widehat{M}$ and $x \in M$. If Q is unital then $\widehat{M}$ is pre-unital or unital according to whether M is pre-unital or unital, respectively. If $k : M \to N$ is a homomorphism of right Q-modules then $\widehat{k} : \widehat{N} \to \widehat{M}$ is a homomorphism of left Q-modules.

Example 2.4 Let L be any sup-lattice such that $L \cong \widehat{L}$ (e.g., $\mathcal{L}(H)$ for some Hilbert space H or 2^X for some set X). Then L is both a right and a left module over $\mathcal{Q}(L)$. For instance, 2^X is a left module over $2^{X \times X}$, with action given by $R \cdot Y = \{x \in X \mid xRy \text{ for some } y \in Y\}$.

We will be interested in working with a specific category of left modules over a variable quantale. Accordingly, by a *pre-unital (left) module* we mean a pair (Q, L) where Q is a unital quantale and L is a pre-unital left Q-module. A module (Q, L) is *unital* if L is unital. A *morphism* of pre-unital modules $(Q, L) \to (Q', L')$ is defined to be a pair (h, k), where $h : Q \to Q'$ is a pre-unital quantale homomorphism and $k : L \to L'$ is a sup-lattice homomorphism such that $k(\top_L) = \top_{L'}$ (i.e., k is strong) and $k(a \cdot x) = h(a) \cdot k(x)$ for all $a \in Q$ and $x \in L$. A morphism (h, k) is *unital* if h is unital. We denote the category of pre-unital modules and their morphisms by $\mathbf{QM}$, and its subcategory whose morphisms are unital is denoted by $\mathbf{QM}_1$.

Example 2.5 1. Let Q be a unital quantale. Then $(Q, \mathcal{R}(Q))$ is a unital module. For any pre-unital (resp. unital) quantale homomorphism $h : Q \to Q'$ there is a unique map $k : \mathcal{R}(Q) \to \mathcal{R}(Q')$ such that (h, k) is a pre-unital (resp. unital) morphism. Hence, the full subcategory of $\mathbf{QM}$ (resp. $\mathbf{QM}_1$) whose modules are of this form is isomorphic to $\mathbf{Qu}$ (resp. $\mathbf{Qu}_1$).

2. Let G be a set. The module $(2^{G^*}, \mathcal{R}(2^{G^*}))$ is freely generated by G in $\mathbf{QM}_1$, in the sense that for any pre-unital module (Q, L) and map $f : G \to Q$ there is a unique unital morphism $h^\sharp : (2^{G^*}, \mathcal{R}(2^{G^*})) \to (Q, L)$ whose quantale part $2^{G^*} \to Q$ extends h.

Let G be a set, and $R, S \subseteq 2^{G^*} \times 2^{G^*}$. We denote by $QM_1\langle G \mid R; S\rangle$ the quotient in $\mathbf{QM}_1$ of the free module $(2^{G^*}, \mathcal{R}(2^{G^*}))$ by the (two-sorted) congruence generated by (R, S'), where S' is the set of those pairs of the form $(X \cdot G^*, Y \cdot G^*)$ with $(X, Y) \in S$. We say $QM_1\langle G \mid R; S\rangle$ is the module *presented* (in $\mathbf{QM}_1$) with *generators* G, *quantale relations* R and *left module relations* S.

3. Finite observations, quantales and systems

This section introduces tropological systems. We start with some motivation about finite observations in §3.1. Then in §3.2, under the assumption that finite observations form quantales, we define systems as being certain quantale representations. In §3.3 we discuss observable properties of systems and conclude that they should be described by left quantale modules. In §3.4 we refine the definition of system so as to take observable properties into account.

3.1. MOTIVATION

There are many kinds of systems and many ways of observing them. Loosely speaking, by a *finite observation* we mean some procedure, or class of "equivalent" procedures, whose execution has finite duration and in the course of which a finite amount of well defined information is exchanged between an observer and a system. Examples of finite observations are: watching on a computer screen that a particular action is enabled; pressing a key of a computer keyboard and watching the subsequent output on the screen for a while; firing a bullet at a wall; measuring the spin of an electron along a specific axis.

Most of these observations are accompanied by internal changes in the system being observed. For instance, pressing "Enter" on a keyboard may cause a computation to start and some internal registers to be updated; and measuring the spin of an electron along the z-axis is understood to project the spin into one of the two spin eigenstates along that axis.

Under this very general definition of finite observation, some natural structure arises:

- Given two finite observations a and b there is a third one, denoted by $a \cdot b$, which consists of performing a and then b; accordingly, $a \cdot b$ is read "a and then b".
- Given a collection $\{a_i\}_{i \in I}$, we can think of each a_i as being a particular way of performing a more general finite observation, which we denote by $\bigvee_{i \in I} a_i$ or $\bigvee\{a_i \mid i \in I\}$, and refer to as being the *disjunction* of the a_i's. When $I = \{1, 2\}$ we write $a_1 \vee a_2$.
- We can simply do nothing and obtain no information whatsoever. Such an observation will be denoted by 1.

The main argument for why a disjunction of finite observations should still be a finite observation is that it is performed by observing one of the disjuncts, which are all finite. This is the interpretation in [Abramsky and Vickers 1993; Resende 2000; Resende 1999a; Resende 1999b], but a word of caution is in order. For instance, measuring the energy of a hydrogen atom by means of the observation $(E = E_1) \vee (E = E_2)$ can mean that we performed a measurement that yielded $E \in \{E_1, E_2\}$; it is reasonable to suppose that the resulting state can be any linear combination of the eigenstates corresponding to E_1 and E_2, which is certainly not the case if $(E = E_1) \vee (E = E_2)$ is interpreted to mean that the energy was measured and the result was E_1 or E_2, but we do not say which. In fact, the distinction between these two kinds of disjunction is the motivation behind the notion of operational resolution of [Coecke and Stubbe 1999a; Coecke and Stubbe 1999b]. Regardless of the interpretation in mind, we will take disjunctions of finite observations to be finite.

3.2. SYSTEMS

The kind of argumentation above leads to the dogma that for any system the set of finite observations that we can perform on it has the structure of a unital quantale whose multiplication means "and then", whose unit is the null observation and whose joins are disjunctions.

From here on we assume that systems have internal states and that performing observations can lead from some states to other states. Accordingly, we define:

Definition 3.1 A *system* consists of a pair (M, Q), where Q is a unital quantale, and M is a pre-unital right Q-module which is atomic[2] as a complete lattice. If the module is unital the system is said to be *stable*, otherwise it is *unstable*. The atoms of M are called *states* and the elements of Q are the *finite observations*. The set of states of (M, Q) is denoted by $\mathcal{S}(M)$. We say that a system is *classical* if M is order-isomorphic to 2^P for some set P, and *quantum* if M is order-isomorphic to the lattice of closed linear subspaces $\mathcal{L}(H)$ of a Hilbert space H.

Definition 3.2 Let (M, Q) be a system, $a \in Q$ and p, q two states. We write $p \xrightarrow{a} q$ if $q \leq p \cdot a$. The relation $\xrightarrow{a}$ is a binary relation on the set of states and it is called the *transition relation* (of a).

The intuition behind the previous definition is that $p \xrightarrow{a} q$ should mean that if the system is at state p then the observation a can be made and if it is made the state after the observation is finished can be q. It is easy to see that the transition relations have the following properties:

1. $p \xrightarrow{1} p$ for all p, or $p \xrightarrow{1} q \Leftrightarrow p = q$ if the system is stable; thus an unstable system can undergo changes of state even when we do not observe it;

2. $p \xrightarrow{a \cdot b} q$ if there exists a state r such that $p \xrightarrow{a} r \xrightarrow{b} q$.

3. $p \xrightarrow{\bigvee S} q$ if and only if there is a family $\{a_i\}_{i \in I}$ of observations of S and a family $\{q_i\}_{i \in I}$ of states such that $p \xrightarrow{a_i} q_i$ for all $i \in I$ and $q \leq \bigvee_{i \in I} q_i$.

[2]By *atomic* we mean that any element in the complete lattice is a join of atoms. Some authors call such a complete lattice *atomistic*.

In the case when the system is classical it is easy to see that condition 2 can be replaced by an equivalence and that condition 3 can be replaced by the simpler

4. $p \stackrel{\bigvee S}{\to} q$ if and only if there exists $a \in S$ such that $p \stackrel{a}{\to} q$.

In fact, in the case of a classical system the assignment $a \mapsto \stackrel{a}{\to}$ defines a pre-unital quantale homomorphism $Q \to 2^{P\times P}$ (unital if and only if the system is stable), where P is the set of states. The system can be recovered from this homomorphism because the module M is order-isomorphic to 2^P and there is a unital isomorphism of quantales $2^{P\times P} \cong \mathcal{Q}(2^P)$. Hence, a classical system can be identified with a pre-unital quantale homomorphism $Q \to 2^{P\times P}$ or, equivalently, with a structure $(P, Q, \{\stackrel{a}{\to}\}_{a\in Q})$ where P is a set, Q is a unital quantale and for each $a \in Q$ we have $\stackrel{a}{\to} \subseteq P \times P$ in such a way that the conditions 1, 2 and 4 are satisfied.

3.3. OBSERVABLE PROPERTIES

In the applications of §§ 4–5 we will see that the algebraic structure of finite observations is not enough to study all that we need about systems because it is also necessary to take into account their observable properties. The aim of the present section is to show that the set of such properties of a system is a left module over its quantale of finite observations.

Let (M, Q) be a system. A *finitely observable property* of the system is anything that can be asserted about the system by performing a finite observation. Of course, at different states the system has different properties, and as a first approximation we can identify the finitely observable properties with the finite observations themselves, saying that a system at the state p has the property a if and only if a can be observed at the state p, i.e., if and only if $p \stackrel{a}{\to} q$ for some state q. Due to the atomicity of M this condition is equivalent to $p \cdot a \neq 0$, and we abbreviate it by writing just $p \stackrel{a}{\to}$. The following conditions hold:

1. $p \stackrel{1}{\to}$ and $p \stackrel{\top}{\to}$ for all states p;
2. $p \stackrel{a\cdot b}{\to}$ if and only if $p \stackrel{a}{\to} q \stackrel{b}{\to}$ for some state q;
3. $p \stackrel{\bigvee S}{\to}$ if and only if $p \stackrel{a}{\to}$ for some $a \in S$;
4. for all $X, Y \subseteq \mathcal{S}(M)$, if $\bigvee X \leq \bigvee Y$ and $p \stackrel{a}{\to}$ for some $p \in X$ then there exists $q \in Y$ such that $q \stackrel{a}{\to}$.

Condition 3 is a consequence of the following equivalences:

$$p \cdot (\bigvee S) \neq 0 \iff \bigvee\{p \cdot a \mid a \in S\} \neq 0 \iff \exists_{a\in S}(p \cdot a \neq 0)\ .$$

Condition 4 follows from the fact that for all $Z \subseteq \mathcal{S}(M)$ we have $(\bigvee Z) \cdot a \neq 0$ if and only if $r \stackrel{a}{\to}$ for some $r \in Z$.

Of course, it may happen that two different finite observations should really be regarded as the same property. For instance, this is the case with the observations 1 and $\top$, which can both be performed in every state, so they distinguish no states. In general, two finite observations a and b represent the same property if and only if for all states x we have $x \xrightarrow{a}$ if and only if $x \xrightarrow{b}$, in which case we write $a \equiv b$. Clearly, $\equiv$ is an equivalence relation on Q. It is also a congruence for joins of Q, for if $a_i \equiv b_i$ for all $i \in I$ then

$$x \xrightarrow{\bigvee_i a_i} \iff \exists_i (x \xrightarrow{a_i}) \iff \exists_i (x \xrightarrow{b_i}) \iff x \xrightarrow{\bigvee_i b_i} .$$

However, in general it is not a congruence for multiplication, being instead a congruence for multiplication on the left (that is, if $a \equiv b$ then $c \cdot a \equiv c \cdot b$ for all $c \in Q$), and thus the quotient $Q/\equiv$ is a unital left Q-module. This motivates the idea that left quantale modules are the right setting in which to describe finitely observable properties.

3.4. SYSTEMS REVISITED

We now refine our previous notion of system so as to take properties of systems into account.

Let (M, Q) be a system and L a pre-unital left Q-module (not necessarily a quotient of Q as above) whose elements are supposed to describe the properties of the system, observable or not. A *satisfaction relation* between the system and L is a binary relation $\vDash \subseteq M \times L$ that satisfies the following conditions, for all $x \in M$, $X \subseteq M$, $\varphi \in L$, $\Phi \subseteq L$:

- $x \models \top_L$ if $x \neq 0$;
- $x \models a \cdot \varphi$ if and only if $x \cdot a \models \varphi$;
- $x \models \bigvee \Phi$ if and only if $x \models \psi$ for some $\psi \in \Phi$;
- $X \models \bigvee \Phi$ if and only if $y \models \psi$ for some $y \in X$.

These conditions mimic the properties of the unary $(- \xrightarrow{a})$ relations. In particular, it can be verified that for all states $p \in \mathcal{S}(M)$ one has $p \models a \cdot \varphi$ if and only if $p \xrightarrow{a} q \models \varphi$ for some state $q \in \mathcal{S}(M)$. Satisfaction relations are equivalent to those sup-lattice bimorphisms $f : M \times L \to 2$ such that $f(x, \top_L) = \top$ for all $x \neq 0$ and which are "middle-linear" in the sense that $f(x \cdot a, \varphi) = f(x, a \cdot \varphi)$, which is equivalent to the second condition. Hence, they can be further identified with certain sup-lattice homomorphisms $M \otimes L \to 2$, and observing that

$$\mathbf{SL}(M \otimes L, 2) \cong \mathbf{SL}(L \otimes M, 2) \cong \mathbf{SL}(L, \mathbf{SL}(M, 2)) = \mathbf{SL}(L, \widehat{M}) ,$$

we conclude that satisfaction relations are also equivalent to strong left Q-module homomorphisms $L \to \widehat{M}$, where being strong is equivalent to the first condition in the definition of satisfaction relation, and the preservation of the action is equivalent to middle-linearity.

Definition 3.3 A *tropological system* $(M, Q, L, \vDash)$ consists of a system (M, Q) together with a pre-unital left Q-module L (of *properties*) and a satisfaction relation $\vDash$.

Proposition 3.4 *Any tropological system $(M, Q, L, \vDash)$ can be identified with a morphism of pre-unital modules $(Q, L) \to (Q, \mathcal{Q}(\widehat{M}))$. The system is stable if and only if the morphism is unital.*

In the case of a classical system the fourth condition in the definition of satisfaction relation is automatically met and thus if the system is also stable we are led to the tropological systems of [Resende 2000], which in this paper will be called *stable classical tropological systems.* They can be presented as unital morphisms $(Q, M) \to (2^{P \times P}, 2^P)$, or equivalently as structures $(P, Q, \{\stackrel{a}{\to}\}_{a \in Q}, L, \vDash)$ where P is the set of states, Q is the unital quantale of finite observations, L is the left Q-module of system properties, and the relations $\stackrel{a}{\to}$ $(a \in Q)$ and $\vDash$ satisfy the following conditions:

- $p \stackrel{1}{\to} q$ if and only if $p = q$,
- $p \stackrel{a \cdot b}{\to} q$ if and only if $p \stackrel{a}{\to} r \stackrel{b}{\to} q$ for some $r \in P$,
- $p \stackrel{\bigvee X}{\to} q$ if and only if $p \stackrel{a}{\to} q$ for some $a \in X$,
- $p \models \top_L$,
- $p \models a \cdot \varphi$ if and only if $p \stackrel{a}{\to} q$ and $q \models \varphi$ for some $q \in P$,
- $p \models \bigvee Y$ if and only if $p \models \varphi$ for some $\varphi \in Y$.

This can be generalized in order to cope with unstable systems simply by replacing the first condition by the weaker $p \stackrel{1}{\to} p$ and allowing the left module L to be pre-unital.

Definition 3.5 Let (Q, L) be a unital module. We say that (Q, L) is *complete* if for all $a, b \in Q$ such that $a \not\leq b$ there exists a stable classical tropological system $(M, Q, L, \vDash)$ with states p, q such that $p \stackrel{a}{\to} q$ and $p \stackrel{b}{\not\to} q$, and for all $\varphi, \psi \in L$ such that $\varphi \not\leq \psi$ there exists a stable classical tropological system $(M, Q, L, \vDash)$ with a state p such that $p \models \varphi$ and $p \not\models \psi$.

Remark 3.6 Obviously, different notions of completeness would arise by changing the kind of tropological system. The completeness in the above definition in the one used in the applications of §5.

Example 3.7 If Ω is a frame, then the module (Ω, Ω) is complete if and only if Ω is spatial in the sense of locale theory.

4. Observational logic

We present here an alternative way of dealing with tropological systems, which in practical applications is often easier to work with. This alternative is based on the fact that quantales and modules can be presented by generators and relations, and it makes explicit that these relations can be seen as formulas of a logic whose models are the systems. In this section we describe one such logic for stable classical systems, which was introduced in [Resende 1999a]. In §4.1 we describe it and in §4.2 an example is discussed.

4.1. OBSERVATIONAL LOGIC FOR STABLE CLASSICAL SYSTEMS

For the purposes of this section of the paper, a system will be understood to be just what in computer science is usually known as a *labelled transition system*:

Definition 4.1 A *labelled transition system* is a structure (P, A, T) such that P is a set (of *states*), A is a set (of *labels*) and T is a map $A \to 2^{P\times P}$. For each $a \in A$ the relation $T(a) \subseteq P \times P$ is the *transition relation of* a and it is denoted by $\stackrel{a}{\to}$. We write $p \stackrel{a}{\to}$ if $p \stackrel{a}{\to} q$ for some $q \in P$.

The idea is that we will forget for a moment about the quantale structure of observations and view a system just as consisting of a black box with which we can interact through specific channels (e.g., lights, buttons, etc.). Each interaction through a channel (e.g., pressing a button) is an example of an *elementary finite observation*. Other finite observations can be obtained from the elementary ones by means of unions and finite concatenations. In other words, the elementary observations are the generators of the quantale of finite observations of a system.

Definition 4.2 Let A be a set (of elementary observations).

1. An *observation formula* over A is a triple $(a, 0, b)$, with $a, b \subseteq A^*$, which we write $a = b$. A *property formula* over A is a triple $(a, 1, b)$, with $a, b \subseteq A^*$, which we write $a =' b$. As abbreviations we also write $a \leq b$ instead of $a \vee b = b$, and $a \leq' b$ instead of $a \vee b =' b$.

2. We define a *satisfaction relation* $\Vdash$ between transition systems labelled over A and (observation and property) formulas over A as follows:

   (a) $(P, A, T) \Vdash a = b$ if for all $p, q \in P$, $p \stackrel{a}{\to} q \Leftrightarrow p \stackrel{b}{\to} q$

   (b) $(P, A, T) \Vdash a =' b$ if for all $p \in P$, $p \stackrel{a}{\to} \Leftrightarrow p \stackrel{b}{\to}$

   When $Sys \Vdash \zeta$ holds we say that the system Sys *satisfies* the formula ζ. A formula is *valid* if it is satisfied by all systems, and we write $\vDash \zeta$.

3. Let Φ be a set of formulas over A and ζ a formula over A. A *model* of Φ is a transition system that satisfies all the formulas of Φ. If every model of Φ

satisfies ζ we say that Φ *entails* ζ, and write $\Phi \vDash \zeta$. We denote by $\Phi^{\vDash}$ the set of all the formulas entailed by Φ.

Since subsets of A^* are elements of the quantale 2^{A^*}, we will usually write set theoretic operations using quantale notation, e.g. $a \vee b$ instead of $a \cup b$. We also omit brackets in singletons, e.g. we write $t \vee u$ instead of $\{t\} \vee \{u\}$ $(= \{t, u\})$.

The connection to (stable and classical) tropological systems is as follows. Let Φ be a set of formulas over a set A. Define sets $R(\Phi), S(\Phi) \subseteq 2^{A^*} \times 2^{A^*}$ by $R(\Phi) = \{(a,b) \mid (a,0,b) \in \Phi\}$ and $S(\Phi) = \{(a,b) \mid (a,1,b) \in \Phi\}$. Then (P,A,T) satisfies Φ if and only if the unique unital morphism $(2^{A^*}, \mathcal{R}(2^{A^*})) \to (2^{P\times P}, 2^P)$ whose quantale part extends T factors through the quotient $QM_1\langle A \mid R(\Phi); S(\Phi)\rangle$. Hence,

Proposition 4.3 *There is a bijection between models (P,A,T) of Φ and classical stable tropological systems whose left module is $QM_1\langle A \mid R(\Phi); S(\Phi)\rangle$ and whose set of states is P.*

Definition 4.4 Let A be a set. We define the following *inference rules over A* $(a, b, a', \ldots \in 2^{A^*})$:

$$\mathbf{Q}: \ \frac{}{a \le b} \ (\text{if } a \subseteq b) \qquad \mathbf{QS}: \ \frac{a = b}{b = a} \qquad \mathbf{QT}: \ \frac{a = b \quad b = c}{a = c}$$

$$\mathbf{Q\bullet}: \ \frac{a = a' \quad b = b'}{a \cdot b = a' \cdot b'} \qquad \mathbf{Q}\bigvee: \ \frac{\{a_i = b_i\}_{i\in I}}{\bigvee_{i\in I} a_i = \bigvee_{i\in I} b_i}$$

$$\mathbf{I}: \ \frac{a \le 1}{a \cdot a = a} \qquad \mathbf{C}: \ \frac{a \le 1 \quad b \le 1}{a \cdot b = b \cdot a}$$

$$\mathbf{PP}: \ \frac{c \le 1 \quad c' \le 1 \quad c \cdot a \cdot c' = a \quad d \le 1 \quad d' \le 1 \quad d \cdot a \cdot d' = a}{c \cdot d \ a \cdot c' \cdot d' = a}$$

$$\mathbf{M}: \ \frac{a = b}{a =' b} \qquad \mathbf{MS}: \ \frac{a =' b}{b =' a} \qquad \mathbf{MT}: \ \frac{a =' b \quad b =' c}{a =' c}$$

$$\mathbf{M\bullet}: \ \frac{b =' b'}{a \cdot b =' a \cdot b'} \qquad \mathbf{M}\bigvee: \ \frac{\{a_i =' b_i\}_{i\in I}}{\bigvee_{i\in I} a_i =' \bigvee_{i\in I} b_i}$$

$$\mathbf{Top}: \ \frac{}{a \le' 1} \qquad \mathbf{W}: \ \frac{c \le 1 \quad a \le' c}{a = c \cdot a}$$

$$\mathbf{M\vee\bullet}: \ \frac{c \le 1 \quad a \le' c \vee b}{a \le' c \cdot a \vee b}$$

If Φ is a set of formulas over A we denote by $\Phi^{\vdash}$ the closure of Φ under all the above rules, and write $\Phi \vdash \zeta$ if $\zeta \in \Phi^{\vdash}$.

The above definition gives us a proof system with infinitary inference rules, where $\Phi \vdash \zeta$ means that ζ can be derived from Φ by means of a possibly infinite derivation.

Theorem 4.5 (Soundness) *If* $\Phi \vdash \zeta$ *then* $\Phi \models \zeta$.

Some of the inference rules, e.g., **QS**, **QT**, **Q•** and **Q**$\bigvee$, are essentially rules of (infinitary) equational logic. Others are specific to observational logic. For instance, **I** (idempotency) means that any system that satisfies $a \leq 1$ also satisfies $a \cdot a = a$. The soundness of this rule follows easily from the fact that any (stable) system which satisfies $a \leq 1$ must be such that $p \xrightarrow{a} q \Rightarrow p = q$ for all states p, q.

Theorem 4.6 (Weak completeness) *If* $\emptyset \models \zeta$ *then* $\emptyset \vdash \zeta$.

This completeness result only needs the "equational part" of the proof system. On the other hand, even with the whole system the logic is not strongly complete; that is, we may have $\Phi^{\vdash} \neq \Phi^{\models}$ for some sets Φ. This happens precisely when the module $QM_1\langle A \mid R(\Phi^{\vdash}); S(\Phi^{\vdash})\rangle$ that we obtain from the set $\Phi^{\vdash}$ as in Proposition 4.3 is not complete in the sense of Definition 3.5.

4.2. EXAMPLE: OBSERVING TIME AND SPACE

In this section we give an example of how time can be described using observational logic. For that purpose let us assume that we can make finite observations like "the current time instant is less than q" and "the current time instant is greater than q", for each rational number q. For instance, these could be performed simply by looking at a watch. Let us represent these observations respectively by the symbols "$(-\infty, q)$" and "$(q, +\infty)$". These are just uninterpreted symbols, but intuitively they can be thought of as intervals of real numbers. The way they relate to each other is expressed by the following formulas of observational logic, which are meant to mimic the construction of real numbers as Dedekind sections (see also [Vickers 1996]), and where $q, r \in \mathbb{Q}$.

R1 $1 = \bigvee_q (q, +\infty)$,

R2 $(r, +\infty) \leq (q, +\infty)$ if $q < r$,

R3 $(q, +\infty) = \bigvee_{q<r} (r, +\infty)$,

R4 $1 = \bigvee_q (-\infty, q)$,

R5 $(-\infty, r) \leq (-\infty, q)$ if $q > r$,

R6 $(-\infty, q) = \bigvee_{q>r} (-\infty, r)$,

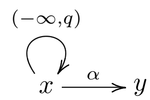
R7 $(q, +\infty) \cdot (-\infty, q) = 0$,

R8 $1 = (q, +\infty) \vee (-\infty, r)$ if $q < r$.

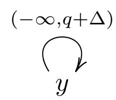
Now let α be another finite observation. We can for example specify that its duration is less than a rational time interval Δ by saying that, for all $q \in \mathbb{Q}$, if the current time instant is less than q then after α is performed the time instant will be less than $q + \Delta$, where the assertion "if the current time instant is less than q then after α is performed the time instant will be less than $q + \Delta$" can be expressed in observational logic by the formula

$$(-\infty, q) \cdot \alpha = (-\infty, q) \cdot \alpha \cdot (-\infty, q + \Delta) \ . \tag{1}$$

In order to understand this formula, notice that any system that satisfies **R4** also satisfies $(-\infty, q) \leq 1$. This means that any states x and y such that $x \stackrel{(-\infty,q)}{\rightarrow} y$ must in fact be equal. Hence, saying that a system satisfies (1) means that it cannot have transitions



without also having



.

Similarly, we can specify that the execution of α lasts for *at least* Δ by means of formulas

$$(q, +\infty) \cdot \alpha = (q, +\infty) \cdot \alpha \cdot (q + \Delta, +\infty) \tag{2}$$

with $q \in \mathbb{Q}$. And it is also possible to specify that α lasts for at most Δ, or at least Δ, when Δ is any real number, respectively by means of the sets of formulas of the form

$$\begin{aligned} (-\infty, q) \cdot \alpha &= (-\infty, q) \cdot \alpha \cdot (-\infty, q + r) \qquad (q, r \in \mathbb{Q},\ r > t) && (3) \\ (q, +\infty) \cdot \alpha &= (q, +\infty) \cdot \alpha \cdot (q + r, +\infty) \qquad (q, r \in \mathbb{Q},\ r < t) && (4) \end{aligned}$$

Any labelled transition system (P, A, T) that satisfies the formulas **R1**–**R8** has a topology on its set of states induced by a subbasis whose open sets are of the form $U(a) = \{p \in P \mid p \stackrel{a}{\rightarrow}\}$, where $a = (-\infty, q)$ or $a = (q, +\infty)$ for some $q \in \mathbb{Q}$. However, if we specify nothing else then we can have unsatisfactory models of time, e.g. with a single time instant! In order to "generate" time we can select some action *tick* (the "tick" of a clock) whose duration is specified by means of formulas like (1)–(4), and add the new property formula $1 \leq' \mathit{tick}$. The latter formula is satisfied if and only if *tick* can be observed at any state, because every state p of any system satisfies $p \stackrel{1}{\rightarrow}$, and also because the formula $1 \leq' \mathit{tick}$ is satisfied if and only if for every state p one has $p \stackrel{1}{\rightarrow} \Rightarrow p \stackrel{\mathit{tick}}{\rightarrow}$. For instance, if the duration of *tick* is specified to be exactly 1, any model (P, A, T) with $P \neq \emptyset$ has at least a countable number of states.

If we want to generate continuous time we can use a generous supply of *tick*'s, e.g. indexed by non-negative reals, possibly supplemented by formulas like

$$\begin{aligned} tick_t \cdot tick_u &= tick_{t+u} \\ tick_0 &= 1 \end{aligned}$$

for all $t, u \in \mathbb{R}$, where the duration of $tick_t$ is specified by observation formulas so as to be exactly t.

In the same way that we have been dealing with time we may also deal with space, by introducing elementary observations like $(-\infty, q)_x$ and $(q, +\infty)_x$, where observing $(-\infty, q)_x$ means that the x-position is less than q, etc. For instance, specifying that for all $t < 0$ the position is $x = 0$ may be accomplished by means of observation formulas $(-\infty, 0) \leq (-\infty, q)_x \cdot (-q, +\infty)_x$ for all rationals q, and specifying that for all $t > 0$ the position is $x = 1$ may be accomplished by means of observation formulas $(0, +\infty) \leq (-\infty, 1+q)_x \cdot (1-q, +\infty)_x$ for all rationals q. An example of a system that satisfies all these formulas is the graph of the function

$$x(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0, \end{cases}$$

(with any value $x(0)$), seen as a subspace of $\mathbb{R}^2$ with the extensions of the elementary observations $(-\infty, q)$, $(-\infty, q)_x$, etc., generating the subspace topology.

5. Process semantics

In this section we make a brief survey of some of the main results and ideas in [Abramsky and Vickers 1993; Resende 2000; Resende 1999b]. We begin with a brief exposition about interleaving semantics of concurrent processes in §5.1, meant to provide some background for those readers who are not familiar with the subject. In particular we give informal definitions of process, process equivalence, and process semantics. Then in §5.2 we describe the results of [Abramsky and Vickers 1993; Resende 2000]. The presentation here is rather different from that in either of those two papers, as it is meant to highlight the essential ideas, and it is developed around a definition of *observational semantics* (Definition 5.2). In §5.3 we hint at some of the issues that arise when non-interleaving semantics are considered.

5.1. SOME BACKGROUND ON INTERLEAVING SEMANTICS

Informally, a *concurrent* system, as opposed to a *sequential* one, can be defined to be a system whose behaviour consists of several activities that run in parallel, possibly interacting with each other in some way. In computer science, concurrent systems appear for example in the form of multiprocessor systems communicating through shared memory, spatially distributed systems whose components communicate through specific channels, networks of mobile phones, hardware devices, etc. The study of concurrent systems is often associated to that of *reactive systems* [Pnueli 1985], which are those that interact repeatedly with their environment and whose behaviour at each instant may depend on the information they have received from the environment. Examples of such systems are operating systems, control circuits in an airplane, etc.

Mathematical models of concurrent and reactive systems are often based on labelled transition systems (see Definition 4.1). This is notably the case with the process algebraic models like CCS [Milner 1980; Milner 1989], CSP [Hoare 1985], ACP [Bergstra and Klop 1984; Baeten and Weijland 1990], or more recently the π-calculus [Milner 1999], the fusion calculus [Parrow and Victor 1998], etc. These models have in common the fact that a *language* is used for describing concurrent systems. The behaviour of the systems is then described by specifying a set of triples of the form $E \xrightarrow{\alpha} F$, where E and F are expressions describing systems and α is a possible action. A triple like $E \xrightarrow{\alpha} F$ specifies that the system described by the expression E can execute the action α and then its behaviour can become that of the system that is described by the expression F. Such a set of triples defines an *operational semantics* for the language. In other words, such a semantics consists of a labelled transition system whose set of states is just the set of all expressions that denote systems. This often leads to identifying the notions of system and state, for in a transition $E \xrightarrow{\alpha} F$ the system F can be thought of as a state of the system E, of which E itself is the initial state.

For instance, in the CCS language examples of expressions are 0, denoting a system that can do no action, $\alpha.E$, denoting a system whose behaviour consists of doing α and then behaving like E, or $E|F$, whose behaviour is that of E and F viewed together as a single system, possible with some communication between E and F. More precisely, the transitions of $E|F$ can be obtained in one of the three following ways:

- $E|F \xrightarrow{\alpha} E'|F$, where $E \xrightarrow{\alpha} E'$, or
- $E|F \xrightarrow{\alpha} E|F'$, where $F \xrightarrow{\alpha} F'$, or
- $E|F \xrightarrow{\tau} E'|F'$, where $E \xrightarrow{\beta} E'$ and $F \xrightarrow{\bar{\beta}} F'$ for some $\beta \neq \tau$.

In the third case β and $\bar{\beta}$ are complementary actions of the same type (e.g., an input and an output) that may interact and produce a "hidden action" τ (e.g., the output sends a signal through the input, and as a result something happens to the two systems). With the exception of the third case, in which the executions of β and $\bar{\beta}$ are synchronized, in the two others only one of the systems changes its state. In other words, parallel execution of actions is being modeled by means of interleaved execution of actions from each component, which motivates the name "interleaving semantics".

The next step in defining a semantics for systems based on labelled transition systems is the recognition that two distinct expressions may denote two systems that should be considered equivalent for all practical purposes (i.e., replacing one by the other within a larger system should produce no *observable* difference).

The main difficulty then is in determining when two such expressions should be considered equivalent. While the problem for classical algorithms is easy (i.e., two algorithms are equivalent if they compute the same function—at least if we disregard considerations about efficiency), for concurrent and reactive systems it became obvious after some time that different notions of equivalence of systems should depend on which observations are understood to be performable, and this depends on each specific situation. Most of these notions of equivalence can be formulated on an arbitrary labelled transition system, without any regard for the syntactical structure of its states. An equivalence class of states is often called a *process*, and the equivalence relations themselves are usually referred to as *process equivalences*. Informally, a process is then the observable behaviour of a system at some state. For a comparative study of many process equivalences see [Glabbeek 1990; Glabbeek 1993]. As examples we mention here *trace equivalence* and *strong bisimulation*, which are usually regarded respectively as the coarsest (i.e., distinguishing fewer systems) and the finest process equivalences one would like to consider. Let p, q be two states of a labelled transition system (P, Act, T). We say p and q are *trace equivalent*, and write $p \sim_T q$, when exactly the same finite sequences of actions can be executed from p and q. We say p and q are *strongly bisimilar*, and write $p \sim_B q$, when there exists a *strong bisimulation relation* R on P such that pRq, where $R \subseteq P \times P$ is a *strong bisimulation relation* if for all states $x, y, x', y' \in P$ and all actions $\alpha \in Act$ we have

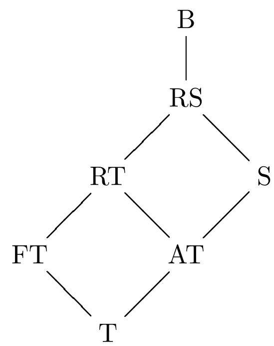
$$\begin{array}{rcl} xRy,\ x \stackrel{\alpha}{\to} x' & \Rightarrow & \exists_{y''}(y \stackrel{\alpha}{\to} y'',\ x'Ry'')\ , \\ xRy,\ y \stackrel{\alpha}{\to} y' & \Rightarrow & \exists_{x''}(x \stackrel{\alpha}{\to} x'',\ x''Ry')\ . \end{array}$$

It is simple to see that $p \sim_B q \Rightarrow p \sim_T q$ for all states $p, q \in P$.

By a *process semantics* we mean a map that to each labelled transition system assigns a process equivalence on its set of states. Trace equivalence and strong bisimulation, along with the usual process equivalences in the literature, are defined on arbitrary labelled transition systems in a uniform manner, and thus they are examples of process semantics.

5.2. OBSERVATIONAL SEMANTICS

In [Abramsky and Vickers 1993] and [Resende 2000] several process semantics are described by means of quantales and modules, namely trace equivalence (T), acceptance-trace equivalence (AT), failure-trace equivalence (FT), ready-trace equivalence (RT), simulation (S), ready-simulation (RS) and bisimulation (B). These form the following lattice, where finer equivalences (i.e., that distinguish more states) are placed above coarser ones:



All the above process equivalences except B are addressed in [Abramsky and Vickers 1993], and B is handled in [Resende 2000]. We refer the reader to [Abramsky and Vickers 1993] for a list of definitions and references concerning the above equivalences.

In this section we treat stable classical tropological systems in the form described after Proposition 3.4.

Definition 5.1 Let $(P, Q, \{\xrightarrow{a}\}_{a\in Q}, L, \models)$ be a stable classical tropological system. Two states $p, q \in P$ are *behaviourally equivalent*, and write $p \sim q$, if for all $a \in Q$ we have $p \xrightarrow{a}$ if and only if $q \xrightarrow{b}$.

The key idea in [Abramsky and Vickers 1993; Resende 2000] is the following. There is a set Act of actions, where actions can be thought of as buttons on a black box that also coincide with the labels of a labelled transition system (P, Act, T). Pressing a button (i.e., executing an action) is an observable action. However, not all observations are like that. For instance, we may regard a failed attempt to press a blocked button α as another observation $\alpha^\times$. Clearly, this is not the same as pressing a button, as in particular such a failed attempt is unlikely to change the state of the system. Another example is watching on a menu that α is a possible action, in which case we know that the corresponding button will not be blocked if we try to press it. Hence, the actions in Act are some of the possible observations on a system, but alone they are not capable of generating the whole quantale of observations. Instead the labelled transition system (P, Act, T) we started with should be regarded as an incomplete presentation of a stable classical tropological system $(P, Q, \{\xrightarrow{a}\}_{a\in Q}, L, \models)$. For instance, if we wish to consider that the failure $\alpha^\times$ mentioned above is a possible observation, then we expect to have $\alpha^\times \leq 1$ in Q, meaning that (at least for stable systems) observing $\alpha^\times$ does not change the state, and $\alpha^\times \cdot \alpha = 0$, for it should be impossible to observe a failure $\alpha^\times$ and then press α. The quantale Q_{FT} for FT (see below) is in fact presented with such generators in [Abramsky and Vickers 1993]. We also expect to have $(\alpha^\times \vee \alpha) \cdot \top_L = \top_L$ in L, for in every state either $\alpha^\times$ or α should be observable (equivalently, the property formula $\alpha^\times \vee \alpha =' 1$ of observational logic should be satisfied).

Definition 5.2 Let $S = (P, Act, T)$ be a labelled transition system. Let also Q be a unital quantale and $i : Act \to Q$ a map. A *(stable classical) tropological extension* of S *along* i is a stable classical tropological system $\mathcal{O}(S) = (P, Q, \{\stackrel{a}{\to}\}_{a \in Q}, L, \models)$ such that

- L is a left Q-module quotient of $Q \cdot \top$,
- $p \stackrel{\alpha}{\to} q$ in S if and only if $p \stackrel{i(\alpha)}{\to} q$ in $\mathcal{O}(S)$, for all states $p, q \in P$ and all $\alpha \in Act$.

An *observational semantics* for labelled transition systems over Act is a tuple $(Q, L, i, \mathcal{O})$ such that Q is a unital quantale, L is a left Q-module quotient of $Q \cdot \top$, $i : Act \to Q$ is a map, and $\mathcal{O}$ is a (large) map that assigns to each labelled transition system over Act an extension of it along i whose left module is L. If such an extension is always unique the observational semantics is said to be *strict*. In that case it can be identified with the triple (Q, L, i).

Any observational semantics defines a process semantics that assigns to each labelled transition system the process equivalence induced by its stable classical tropological extension in the manner of Definition 5.1. Many of the results in [Abramsky and Vickers 1993] consist of showing that process equivalences in the literature can be obtained in this way, and can be summarized as follows:

Theorem 5.3 (Abramsky and Vickers 1993) *There are observational semantics for T, AT, FT, RT, S, and RS.*

The authors refer to these results as "first completeness theorems". Although they pay no attention to strictness, in fact it can be shown that their observational semantics are strict. Notice that strictness is a desirable property of any observational semantics because it shows that we can obtain a process semantics in a purely algebraic way (i.e., defining a quantale and a module and the way in which Act is mapped into the quantale), without having to choose an extension for each transition system. In other words, a strict observational semantics is very much the same as defining a set of formulas in observational logic. The authors also obtain similar results for three other process semantics, namely acceptance equivalence, failures equivalence and readiness equivalence, but these fall out of our framework because they use quantaloids instead of just quantales. We conjecture that these three process semantics can be construed as strict observational semantics.

Theorem 5.4 (Resende 2000) *There is a strict observational semantics for B.*

For each of the process equivalences $E \in \{\mathrm{T}, \mathrm{AT}, \mathrm{FT}, \mathrm{RT}, \mathrm{S}, \mathrm{RS}, \mathrm{B}\}$, let $\mathcal{M}_E$ be the module (in $\mathbf{QM}_1$) of the corresponding observational semantics. The results in [Abramsky and Vickers 1993] (for T, AT, FT, RT, S and RS) and [Resende 2000] (for B) also enable us to prove:

Theorem 5.5 *The modules $\mathcal{M}_E$, for $E \in \{\mathrm{T}, \mathrm{AT}, \mathrm{FT}, \mathrm{RT}, \mathrm{S}, \mathrm{RS}, \mathrm{B}\}$, are complete with respect to stable classical tropological systems.*

In [Abramsky and Vickers 1993] these completeness results are called "second and third completeness theorems".

Remark 5.6 Beware that the notation we are using can be misleading. The modules $\mathcal{M}_E$ are denoted in [Abramsky and Vickers 1993] and [Resende 2000] as follows:

- $\mathcal{M}_E = (Q_E, Q'_E)$ for $E \in \{\mathrm{T}, \mathrm{AT}, \mathrm{FT}, \mathrm{RT}\}$
- $\mathcal{M}_{\mathrm{RS}} = (Q_{\mathrm{hf}}, Q'_{\mathrm{hf}})$
- $\mathcal{M}_{\mathrm{B}} = (Q_{\mathrm{B}}, \Omega_{\mathrm{B}})$
- $\mathcal{M}_{\mathrm{S}}$ is not mentioned explicitly.

There are also modules $(Q_{\mathrm{RS}}, Q'_{\mathrm{RS}})$ and $(Q_{\mathrm{S}}, Q'_{\mathrm{S}})$ in [Abramsky and Vickers 1993], but with a different meaning (see below).

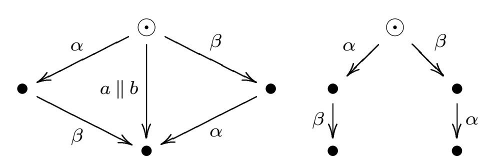
In [Abramsky and Vickers 1993] it is argued that the quantales obtained for simulation and ready-simulation contain generators that represent observations whose feasibility in practice is not clear; that is, it is not clear which experimental setup would enable us to perform such observations. Hence, the authors construct new quantales, respectively Q_{S} and Q_{RS}, with more "realistic" generators, and corresponding modules Q'_{S} and Q'_{RS}. For instance, their quantale Q_{hf} (see the above remark) is a subquantale of Q_{RS}; in Q_{hf} there are generators $\alpha^{\times}$ for failures as discussed before Definition 5.2, which in Q_{RS} can be defined by $\alpha^{\times} = \alpha \cdot \natural$, where $\natural$ represents the action of pressing an "undo" button that goes back to the previous state before α was executed. However, this requires the set of states to be expanded, too; that is, given a labelled transition system its extension as a tropological system must contain a larger set of states, each of which can be seen as a particular way of realizing one of the original states, in the sense that it is behaviourally equivalent to it if we restrict to the finite observations in the subquantales without the new generators (e.g. Q_{hf}). This expansion of the set of states is required because it may be possible to reach a state by doing α from many different states, and thus the action of undoing α cannot be well defined unless each state can store information about past states.

For bisimulation the situation is at least as complicated. In [Resende 2000] suitable extensions of modules and sets of states are provided, too, but the new module $(\Xi, \Xi \cdot \top)$ for bisimulation is not complete, and no attempts are made to present a complete quotient of it.

5.3. REMARKS ON NON-INTERLEAVING SEMANTICS

There are situations in which it is relevant to take into account that actions have a duration, in which case it no longer makes sense to say that two different actions may only be synchronized or interleaved, for they may also overlap. Also, it may be important to take into account relations of causality or independence between actions, for instance when taking relativity and space-time geometry into account, or simply when certain actions are known to be independent, or causally related, on pragmatic grounds. In order to describe such situations one therefore needs to go beyond interleaving semantics. In concurrency theories this usually involves considering models other than labelled transition systems, such as Petri-nets or event structures. See [Winskel and Nielsen 1995] for definitions of these and other models. See also [Glabbeek and Goltz 1989] for a description of corresponding process equivalences.

There are notions of state associated to such models, e.g. markings for Petri nets, or configurations for event structures, and one can define labelled transition systems on them. In order not to fall back into interleaving semantics, however, it is usually necessary to take as labels more than just the basic actions ("buttons") of a system. An example is provided by the following system, in which $\alpha \parallel \beta$ is the observation that both α and β have occurred and furthermore they were completely independent from each other (because we know they are separated in spacetime, or simply due to pragmatic reasons):



Notice that if we omit $\alpha \parallel \beta$ then the two states marked with "$\odot$" become strongly bisimilar (see §5.1), and thus equivalent with respect to all the usual interleaving semantics. However, the state on the left can represent the joint state of a pair of independent systems, one of which does α and the other does β, whereas the state on the right cannot.

In order to cater for such non-interleaving semantics (also known as *causal semantics* or as "true concurrency") a possibility is to incorporate observations such as $\alpha \parallel \beta$, and indeed many others, into a suitable quantale. It is also important to remark that in the context of non-interleaving semantics it is usual to regard pomsets [Pratt 1986] as being observations on concurrent systems, so it is desirable that a quantale framework for non-interleaving semantics be able to take them into account. Some steps in this direction are taken in [Resende 2000], whose results show that in order to reconcile pomsets and the natural "and then" operation on finite observations one is led in a natural way to quantales as the right framework in which to handle observations on at least those (stable classical) concurrent systems which are described by event structures.

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