Canonical Institutions of Behaviour

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Abstract. The concept of behaviour plays a central role in the specification of a considerable number of different kinds of systems. In these settings a "behaviour" is seen as a possible evolution (or life-cycle) of the system, whereas the system itself is considered to be defined by the set of all its possible behaviours.
Examples of this kind of situation are common. Maybe the most well known and studied is that of concurrency theory: a behaviour is e.g. a stream of actions and the system is a process (in this case, a set of streams of actions).
If institutions are used as the way for specifying the systems, then it is customary to start by creating an institution for individual behaviours (where each model corresponds to a possible behaviour) from which the "system institution" - or "institution of behaviour", in our terminology - where each model is a set of behaviours is built.
The new institution is tightly bound to the base institution, sharing signatures and languages. Also, because the models are obtained from the base institution's models, the satisfaction relation is defined in terms of the base satisfaction relation.
In this paper it is shown that the construction of these institutions of behaviour can be carried out in a canonical way. Indeed, the construction does not depend in any way at all on the particular base institution chosen. It is also shown that several institutions presented since the 90's in WADT workshops and elsewhere arise as particular cases of this canonical construction [4, 2, 3, 6–8].
It is hoped that the proposed construction can be used as a shortcut for defining new useful institutions of behaviour.

1 Introduction

This paper emerged from the necessity of abstracting the methodology used in the past when creating a considerable number of different institutions.

More concretely, one of the authors followed an already recurrent pattern (started by the seminal paper [4] and continued in [2, 3, 7, 8]) when trying to create an institution for hybrid systems[1] [6]. As it had always been the case, the construction heavily relied upon the specific case at hand.

The papers went roughly like this: in the first place a suitable notion of behaviour was considered, such as a stream of actions (in the case of concurrency theory), after which an institution for talking about individual behaviours was defined (for instance, temporal logic). Then, and since the initial goal was to talk about systems consisting of sets of behaviours, another institution was defined based upon the initial one.

Later, J. Félix Costa directed him to some previous work that he had done when studying the relation between temporal theories and models of concurrency. We had resorted to a set of categorial tools that included topological theories.

Although the initial focus was in the search for a categorial formalization of the relation between temporal theories and concurrency models, the present work proves to be more general.

In the one hand, it provides a straightforward way of creating what we call "institutions of behaviour". These arise when specifying a system in terms of its individual behaviours (as was the case in [6]).

In the other hand, it is not restricted to temporal theories, although they indeed seem the right candidate when talking about "behaviours".

Another aspect that initially was somewhat overlooked and is now addressed is denotational semantics. In the abstract setting of institutions of behaviour, the relation between specifications (and, later on, theories) and models is investigated. In particular, it is shown that it is possible to give denotational semantics to the usual constructions of unconstrained and constrained parallel composition (provided these constructions can be represented at the signature level).

We assume that the reader has a working knowledge of the basic notions of category theory, logic and institutions. For the sake of completeness, most of the categorial constructions used are defined

[1] A *hybrid system* is a system exhibiting both discrete and continuous behaviour.

in the paper (a very complete reference can be found in [1]; another good source is [5]).

In Section 2 we present the plan for our journey. We start by defining both the starting and the destination points - behaviour structures and institutions of behaviour, respectively.

The journey itself is conducted in Section 3. Here all the steps to follow in order to transform a behaviour structure into an institution are detailed. The transformation resorts to established categorial tools, which happens to be quite enlightening.

Finally, in Section 4 we tackle the denotational semantics aspects of our quest. We start by relating specifications and models, and then go on to relate theories both with specifications and models. As we'll see, in this framework theories and models appear unified (the unification being materialized by an adjunction).

2 Roadmap

2.1 The ingredients...

The starting point in our journey towards an institution of behaviour is a precise definition of what a behaviour is in the considered context. In our setting, behaviours must be presented via *behaviour structures.*

Definition 1. A *behaviour structure* is a tuple $\langle Sig, Sen, Bh, \Vdash^B \rangle$ where

- Sig is a category whose objects are called *signatures*;
- $Sen : Sig \to Set$ is a functor associating to each signature a set of *sentences*;
- $Bh : Sig^{op} \to Set$ is a functor associating to each signature a set of *behaviours*;
- $\Vdash^B = \{\Vdash^B_\Sigma\}_{\Sigma \in |Sig|}$ is a family of *satisfaction relations* with $\Vdash^B_\Sigma \subseteq Bh(\Sigma) \times Sen(\Sigma)$

and such that for all $\sigma : \Sigma \to \Sigma'$ in Sig, $\varphi \in Sen(\Sigma)$ and $b' \in Bh(\Sigma')$ the following *satisfaction condition* is met:

$$Bh(\sigma)(b') \Vdash^B_\Sigma \varphi \quad \text{iff} \quad b' \Vdash^B_{\Sigma'} Sen(\sigma)(\varphi)$$

Note that behaviour structures closely resemble institutions (indeed they are institutions). We adopted a lighter definition because in our usual setting the added structure of an institution (namely the functor $Int : Sig^{op} \to Cat$ for talking about interpretation structures) is not relevant for describing behaviours.

As an example of a behaviour structure, consider an example from concurrency theory.[2]

Example 2. The behaviour interpretation structure $\mathcal{P}$ of traces consists of the following:

- $Sig = Set$. Each $\Sigma \in |Sig|$ is a set of action symbols;
- for each $\Sigma \in |Sig|$, $Sen(\Sigma)$ is inductively defined as follows:
  - $\Sigma \subseteq Sen(\Sigma)$;
  - $(\neg\varphi), (\mathbf{X}\varphi) \in Sen(\Sigma)$ p.t. $\varphi \in Sen(\Sigma)$;
  - $(\varphi \Rightarrow \psi), (\varphi \mathbf{U} \psi) \in Sen(\Sigma)$ p.t. $\varphi, \psi \in Sen(\Sigma)$.
- for each $\sigma : \Sigma \to \Sigma'$ in Sig, $Sen(\sigma) = \underline{\sigma}$ where
  - $\underline{\sigma}(a) = \sigma(a)$ for each $a \in \Sigma$;
  - $\underline{\sigma}((\neg\varphi)) = (\neg\underline{\sigma}(\varphi))$;
  - $\underline{\sigma}((\varphi \Rightarrow \psi)) = (\underline{\sigma}(\varphi) \Rightarrow \underline{\sigma}(\psi))$;
  - etc.
- $Bh : Sig^{op} \to Set$ is the functor defined by:
  - $Bh(\Sigma) = (2^{\Sigma})^{\mathbb{N}_0}$; each behaviour (or stream) is a sequence containing at each point in time the snapshot of actions that occurred at that point;
  - for $\sigma : \Sigma' \to \Sigma$ in Sig^{op} and $b' \in Bh(\Sigma')$, $Bh(\sigma)(b') = \lambda n.\sigma^{-1}(b'(n))$.
- $\Vdash^B$ is defined as usual: for $\Sigma \in |Sig|$, $b \in Bh(\Sigma)$ and $i \in \mathbb{N}_0$,
  - $b \Vdash^B_\Sigma \varphi$ iff $b, 0 \Vdash^B_\Sigma \varphi$;
  - $b, i \Vdash^B_\Sigma a$ iff $a \in b(i)$;
  - $b, i \Vdash^B_\Sigma (\neg\varphi)$ iff not $b, i \Vdash^B_\Sigma \varphi$;
  - etc.

2.2 ... and the cake

Our aim is to obtain an institution from a behaviour structure. First of all, we recall what an institution is.

[2] As we'll see later, from this structure the process institution as presented in [3], for instance, arises.

Definition 3. An *institution* is a tuple $\langle Sig, Sen, Int, \Vdash\rangle$ where

- Sig is a category whose objects are called *signatures*;
- $Sen : Sig \to Set$ is a functor associating to each signature a set of *sentences*;
- $Int : Sig^{op} \to Cat$ is a functor associating to each signature a category of *interpretation structures*;
- $\Vdash = \{\Vdash_\Sigma\}_{\Sigma\in|Sig|}$ is a family of *satisfaction relations* with each $\Vdash_\Sigma \subseteq |Int(\Sigma)| \times Sen(\Sigma)$

and such that for all signature morphism $\sigma : \Sigma \to \Sigma'$, $\varphi \in Sen(\Sigma)$, and $I' \in |Int(\Sigma')|$, the following *satisfaction condition* is met:

$$Int(\sigma)(I') \Vdash_\Sigma \varphi \quad \text{iff} \quad I' \Vdash_{\Sigma'} Sen(\sigma)(\varphi)$$

We'll refer to institutions arising from behaviour structures as *institutions of behaviour.*

2.3 What about the recipe?

In a behaviour institution we're interested in keeping signatures and languages and having sets of behaviours as interpretation structures. Also, we want to extend the behaviour satisfaction relation from behaviours to sets of behaviours in such a way that the satisfaction condition is met.

How should we then go from a structure $\langle Sig, Sen, Bh, \Vdash^B\rangle$ to an institution $\langle Sig, Sen, Int, \Vdash\rangle$? As we'll see in the following section, there is a canonical way of doing this.

Along the way, some categorial tools are needed to ease the baking process. Most of these tools are presented in [1].

3 Canonical Institutions of Behaviour

It was referred earlier that interpretation structures are to be sets of behaviours. There is a categorial construction, *Spa*, that is well suited for representing this.

Definition 4. A *concrete category* over $\mathcal{X}$ is a pair $\langle \mathcal{A}, U\rangle$ where $\mathcal{A}$ is a category and $U : \mathcal{A} \to \mathcal{X}$ is a faithful functor (the underlying or forgetful functor).

Definition 5. Let $T : \mathcal{X} \to \mathit{Set}$ be a functor. $\mathit{Spa}(T)$ is the concrete category over $\mathcal{X}$ in which:

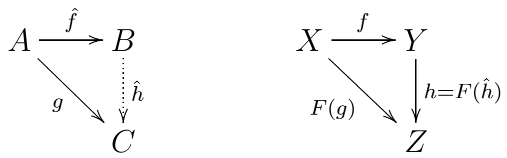
- the objects are pairs $\langle X, \alpha \rangle$ with $X \in |\mathcal{X}|$ and $\alpha \subseteq T(X)$;
- a morphism $f : \langle X, \alpha \rangle \to \langle Y, \beta \rangle$ is a morphism $f : X \to Y$ in $\mathcal{X}$ s.t. $Tf(\alpha) \subseteq \beta$;
- the underlying functor $U : \mathit{Spa}(T) \to \mathcal{X}$ is defined by $U(f : \langle X, \alpha \rangle \to \langle Y, \beta \rangle) = f : X \to Y$.

In the particular case of $\mathit{Spa}(\mathit{Bh})$, objects are pairs $\langle \Sigma, \beta \rangle$ with $\beta \subseteq \mathit{Bh}(\Sigma)$ (that is, β is a set of behaviours) and each morphism $\sigma : \langle \Sigma, \beta \rangle \to \langle \Sigma', \beta' \rangle$ is a Sig^{op} morphism $\sigma : \Sigma \to \Sigma'$ s.t. $\mathit{Bh}(\sigma)(\beta) \subseteq \beta'$ (that is, each behaviour in β must be transformed by $\mathit{Bh}(\sigma)$ into a behaviour of β'). The underlying functor is $bS : \mathit{Spa}(\mathit{Bh}) \to \mathit{Sig}^{op}$.

In order to obtain not only interpretation structures but the complete functor *Int*, a bit more of work is necessary.

Definition 6. Let $F : \mathcal{A} \to \mathcal{X}$ be a functor, $f : X \to Y$ in $\mathcal{X}$ and A in $\mathcal{A}$ s.t. $F(A) = X$. A morphism $\hat{f} : A \to B$ in $\mathcal{A}$ is said to be *cocartesian* by F for f on A if

- $F(\hat{f}) = f$;
- for each pair or morphisms $g : A \to C$ in $\mathcal{A}$ and $h : Y \to Z$ in $\mathcal{X}$ such that $Z = F(C)$ and $h \circ f = F(g)$ there is a unique morphism $\hat{h} : B \to C$ such that $F(\hat{h}) = h$ and $\hat{h} \circ \hat{f} = g$.



Dually, we define cartesian morphisms.

Definition 7. A functor $F : \mathcal{A} \to \mathcal{X}$ is said to be a *cofibration* if for every $f : X \to Y$ in $\mathcal{X}$ and $A \in \mathcal{A}$ such that $F(A) = X$ there is a cocartesian morphism by F for f on A. Dually, we define fibrations.

Definition 8. A *bifibration* is a functor that is simultaneously a fibration and a cofibration.

Proposition 9. In $\mathit{Spa}(T)$, given $f : X \to Y$:

– $f : \langle X, \alpha\rangle \to \langle Y, Tf(\alpha)\rangle$ is cocartesian by U for f on $\langle X, \alpha\rangle$;
– $f : \langle X, (Tf)^{-1}(\beta)\rangle \to \langle Y, \beta\rangle$ is cartesian by U for f on $\langle Y, \beta\rangle$

and so the underlying functor is always a bifibration.

Bifibrations are quite interesting: under some additional conditions, a bifibration $\mathcal{A} \to \mathcal{X}$ induces a functor $\mathcal{X} \to Cat$.

Definition 10. A *cocleavage* κ for a cofibration $F : \mathcal{A} \to \mathcal{X}$ maps each pair $\langle f : F(A) \to Y, A\rangle$ to a cocartesian morphism by F for f on A. κ is said to be a *splitting* if it preserves identity and composition.

Example 11. Consider a functor $T : \mathcal{X} \to Set$ and the underlying functor $U : Spa(T) \to \mathcal{X}$. From Proposition 9 we obtain a cocleavage κ by defining that for each $f : X \to Y$ and $\langle X, \alpha\rangle \in |Spa(T)|$, $\kappa(f, \langle X, \alpha\rangle) = f : \langle X, \alpha\rangle \to \langle Y, Tf(\alpha)\rangle$.

Definition 12. Let $F : \mathcal{A} \to \mathcal{X}$ be a functor and $X \in |\mathcal{X}|$. The *fiber* $\mathcal{A}_X$ of $\mathcal{A}$ over X is the subcategory of $\mathcal{A}$ whose objects are mapped by F to X and whose arrows are mapped to id_X.

We are now ready to go from a functor $\mathcal{A} \to \mathcal{X}$ to another one $\mathcal{X} \to Cat$.

Proposition 13. A cofibration $F : \mathcal{A} \to \mathcal{X}$ with splitting cocleavage κ induces a functor $G : \mathcal{X} \to Cat$ defined by:

– $G(X) = \mathcal{A}_X$;
– for $f : X \to Y$ in $\mathcal{X}$, $G(f) : \mathcal{A}_X \to \mathcal{A}_Y$ is the functor defined by
  - $Gf(A) = cod(\kappa(f, A))$;
  - $Gf(g : A \to A')$ is the only $h : Gf(A) \to Gf(A')$ s.t. $h \circ \kappa(f, A) = \kappa(f, A') \circ g$.

$$\begin{array}{ccc} A & \xrightarrow{\kappa(f,A)} & Gf(A) \\ {\scriptstyle g}\downarrow & \searrow & \vdots\downarrow{\scriptstyle h} \\ A' & \xrightarrow[\kappa(f,A')]{} & Gf(A') \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle id_Y} \\ & & Y \end{array}$$

3.1 Interpretation Structures

Applying this proposition to the particular case of $Spa(T)$ categories provides us with the desired path from Bh to Int.

Corollary 14. In $Spa(T)$ (with $T : \mathcal{X} \to Set$), the underlying functor $U : Spa(T) \to \mathcal{X}$ induces $G : \mathcal{X} \to Cat$ s.t.

- $G(X) = Spa(T)_X$:
  - the objects of $G(X)$ are pairs $\langle X, \alpha \rangle$ with $\alpha \subseteq T(X)$;
  - id_X is a morphism $id_X : \langle X, \alpha \rangle \to \langle X, \beta \rangle$ iff $\alpha \subseteq \beta$.
- for $f : X \to Y$ in $\mathcal{X}$:
  - $Gf(\langle X, \alpha \rangle) = \langle Y, Tf(\alpha) \rangle$;
  - $Gf(id_X) = id_Y$

We thus obtain $Int : Sig^{op} \to Cat$ from $bS : Spa(Bh) \to Sig^{op}$. An interpretation structure is a pair $\langle \Sigma, \beta \rangle$ with $\beta \subseteq Bh(\Sigma)$, and for $\sigma : \Sigma \to \Sigma'$ in Sig^{op},

$$Int(\sigma)(\langle \Sigma, \beta \rangle) = \langle \Sigma', Bh(\sigma)(\beta) \rangle.$$

3.2 Satisfaction

The satisfaction relation for institutions of behaviour must be obtained from the satisfaction relation of the underlying behaviour structure. This is easily accomplished by extending such relation to sets.

Definition 15. For each Σ, $\Vdash_\Sigma \subseteq |Int(\Sigma)| \times Sen(\Sigma)$ is defined by

$$\langle \Sigma, \beta \rangle \Vdash_\Sigma \varphi \quad \text{iff} \quad b \Vdash^B_\Sigma \varphi \text{ for every } b \in \beta$$

The satisfaction condition for the institution of behaviour is now a consequence of the satisfaction condition for the underlying behaviour structure.

Proposition 16. For every $\sigma : \Sigma \to \Sigma'$ in Sig, $\varphi \in Sen(\Sigma)$ and $I' \in |Int(\Sigma')|$:

$$Int(\sigma)(I') \Vdash_\Sigma \varphi \quad \text{iff} \quad I' \Vdash_{\Sigma'} Sen(\sigma)(\varphi)$$

Proof. Let $\sigma : \Sigma \to \Sigma'$ be a morphism in *Sig*, $\varphi \in Sen(\Sigma)$ a formula and $I' = \langle \Sigma', \beta' \rangle$ be a Σ'-interpretation structure. We then have that:

$$\begin{array}{rll} I' \Vdash_{\Sigma'} Sen(\sigma)(\varphi) & \text{iff} & b' \Vdash^{B}_{\Sigma'} Sen(\sigma)(\varphi) \text{ for every } b' \in \beta' \\ & \text{iff} & Bh(\sigma)(b') \Vdash^{B}_{\Sigma} \varphi \text{ for every } b' \in \beta' \\ & \text{iff} & Int(\sigma)(I') \Vdash_{\Sigma} \varphi. \end{array}$$

□

3.3 Institutions of Behaviour

The preceding sections have shown how to go from a behaviour structure to an institution (of behaviour) in a canonical way.

In the particular case of the behaviour structure $\mathcal{P}$, the corresponding institution of behaviour coincides with the institution for processes presented, among others, in [3].

4 Denotational Semantics

Now that we have a canonical way of constructing institutions of behaviour, we want to go a step further and study some of their properties.

This study is heavily biased towards denotational semantics aspects. We are mainly interested in the relation between specifications and interpretation structures, both of each arise from *Spa* constructions: both specifications and interpretation structures are sets (of formulae and behaviours, respectively).

The usefulness of denotational semantics whould be clear: it provides a way of relating constructions in the syntactical and semantic domains. For instance, it is desirable to have the property that putting together specifications corresponds in some way to putting together models of such specifications.

Since (co)products and (co)cartesian liftings are our preferred denotational semantics tools, it is natural to start by determining if (co)products and (co)cartesian liftings are inherent in behaviour institutions. If not, appropriate sufficiente conditions must be sought.

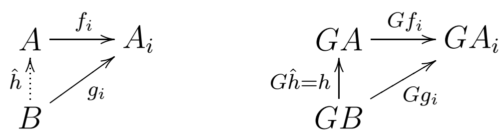
Again, the experience in past papers ([4, 2, 3, 6–8]) is helpful. In all these papers, the existence of (co)products and (co)cartesian liftings is established in an ad-hoc way that can however be extrapolated to a fully general proof: as will be seen, the existence of these constructions is an "almost" intrinsic property of behaviour institutions, depending only on the existence of (co)products in *Sig*.

In the following we'll consider and relate the concrete category $\langle Spa(Sen), sS : Spa(Sen) \to Sig\rangle$ (of specifications) with the concrete category $\langle Spa(Bh), bS : Spa(Bh) \to Sig^{op}\rangle$ (of behaviours).

Again, most of the categorial tools come from [1].

4.1 (Co)completeness et al

Definition 17. Let $G : \mathcal{A} \to \mathcal{B}$ be a functor and $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$ a source in $\mathcal{A}$. $\mathcal{S}$ is called *G-initial* if for each source $\mathcal{T} = (B \xrightarrow{g_i} A_i)_I$ in $\mathcal{A}$ and $h : G(B) \to G(A)$ in $\mathcal{B}$ such that $G\mathcal{T} = G\mathcal{S} \circ h$ there exists a unique $\hat{h} : B \to A$ s.t. $\mathcal{T} = \mathcal{S} \circ \hat{h}$ and $h = G(\hat{h})$.



G is called *topological* if every source $(B \xrightarrow{f_i} GA_i)_I$ has a unique G-initial lift.

Definition 18. *A concrete category $\langle \mathcal{A}, U\rangle$ over $\mathcal{X}$ is said* topological *if U is topological.*

The underlying functor in topological concrete categories has a very nice property, as the following proposition shows.

Proposition 19. Let $\langle \mathcal{A}, U\rangle$ be a concrete category over $\mathcal{X}$. If this concrete category is topological, then U uniquely lifts limits and colimits and preserves limits and colimits.

Proposition 20. Let $T : \mathcal{X} \to Set$ be a functor. In the concrete category $\langle Spa(T), U\rangle$, a source $(\langle X, \alpha\rangle \xrightarrow{f_i} \langle X_i, \alpha_i\rangle)_I$ is U-initial iff $\alpha = \bigcap_I T(f_i)^{-1}(\alpha_i)$, and so each source has a unique U-initial lift.

Additionally, in the concrete category $\langle Spa(T)^{op}, U^{op}\rangle$, a source $(\langle X, \alpha\rangle \xrightarrow{f_i} \langle X_i, \alpha_i\rangle)_I$ is U^{op}-initial iff $\alpha = \bigcup_I T(f_i)(\alpha_i)$.

The preceding proposition shows that in particular every *Spa* category is topological. A deeper result, however, is the following.

Corollary 21. The concrete category $\langle \mathit{Spa}(\mathit{Bh}), \mathit{bS}\rangle$ is (co)complete iff Sig^{op} is (co)complete. Similarly, the category $\langle \mathit{Spa}(\mathit{Sen}), \mathit{sS}\rangle$ is (co)complete iff Sig is (co)complete.

4.2 (Co)products

In the light of our denotational semantics study, the Proposition 19 implies that if (co)products exist in Sig^{op} (resp. Sig), then we have (co)products in $\mathit{Spa}(\mathit{Bh})$ (resp. $\mathit{Spa}(\mathit{Sen})$).

The following proposition characterizes (co)products in $\mathit{Spa}(T)$ categories.

Proposition 22. In the concrete category $\langle \mathit{Spa}(T), U\rangle$ over $\mathcal{X}$:

- if $\langle X, i', i''\rangle$ is a coproduct of X' and X'' in $\mathcal{X}$, then
$$\langle\langle X, Ti'(\alpha') \cup Ti''(\alpha'')\rangle, i', i''\rangle$$
is a coproduct of $\langle X', \alpha'\rangle$ and $\langle X'', \alpha''\rangle$ in $\mathit{Spa}(T)$;
- if $\langle X, p', p''\rangle$ is a product of X' and X'' in $\mathcal{X}$, then
$$\langle\langle X, (Tp')^{-1}(\alpha') \cap (Tp'')^{-1}(\alpha'')\rangle, p', p''\rangle$$
is a product of $\langle X', \alpha'\rangle$ and $\langle X'', \alpha''\rangle$ in $\mathit{Spa}(T)$.

(Co)products are frequently used to model unconstrained parallel composition. Proposition 22 shows how (co)products in the category of signatures can be lifted to (co)products in the concrete categories of specifications and behaviours.

Moreover, the fact that the underlying functors in these two categories are bifibrations enables us to obtain constrained parallel composition by (co)cartesian lifting of an appropriate signature morphism provided the constraints can be expressed at the signature level.[3]

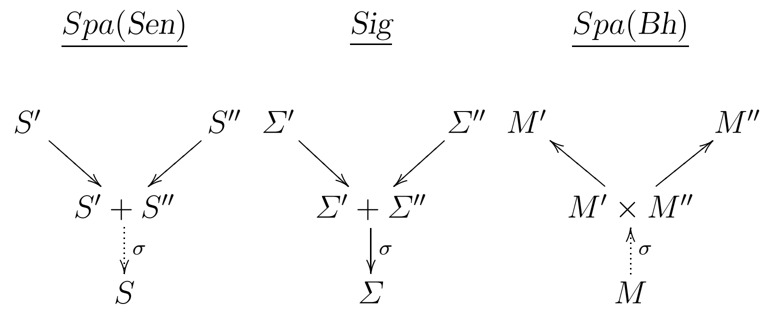
The diagram below illustrates this. Consider two signatures Σ', Σ'', two specifications S', S'' over Σ', Σ'' (respectively) and two interpretation structures M', M'' over Σ', Σ'' (respectively).

Consider also that we want to obtain the constrained parallel composition S (resp. M) of the two specifications (resp. models) using a third signature Σ as the desired final signature. Usually this

[3] As an example using the behaviour structure $\mathcal{P}$, we can represent action sharing or synchronization at the signature level.

signature represents a restriction over the signature of the unconstrained parallel composition.

We can do this first by calculating $\Sigma' + \Sigma''$. Then we must provide a morphism $\sigma : \Sigma' + \Sigma'' \to \Sigma$ expressing the constraints to be observed. Finally, we obtain S and M as (co)cartesian lifting of σ.



Of course, in order to be able to talk about denotational semantics, it is important that the operations carried at the specification level can be transported to the model level. As we'll see in the next section, this is always the case in institutions of behaviour.

4.3 Specifications vs Interpretation Structures

In this section we answer the question: "Is there any relation between $Spa(Sen)^{op}$ and $Spa(Bh)$?". Apart from the trivial observation that both are concrete categories over Sig^{op}, there is a deeper relation: an adjunction $[\![\_]\!] : Spa(Sen)^{op} \to Spa(Bh)$.

Before we proceed we present an adapted version of the "Taut Lift" theorem introduced in [1].

Theorem 23 (Taut lift). Let $\langle \mathcal{A}, U\rangle$ and $\langle \mathcal{B}, V\rangle$ be concrete categories over $\mathcal{X}$, $G : \langle \mathcal{A}, U\rangle \to \langle \mathcal{B}, V\rangle$ a concrete functor and suppose that $\langle \mathcal{A}, U\rangle$ is topological. Then G is (right-)adjoint iff it sends U-initial sources into V-initial sources.

Def./Prop. 24. The functor $[\![\_]\!] : Spa(Sen)^{op} \to Spa(Bh)$ defined by

– $[\![\langle \Sigma, \Phi\rangle]\!] = \langle \Sigma, [\![\Phi]\!]_\Sigma\rangle = \langle \Sigma, \{b \in Bh(\Sigma) \mid b \Vdash^B_\Sigma \Phi\}\rangle$[4];

[4] As usual, $b \Vdash^B_\Sigma \Phi$ means $b \Vdash^B_\Sigma \varphi$ for every $\varphi \in \Phi$

– $[\![\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle]\!] = \sigma : \langle \Sigma, [\![\Phi]\!]_{\Sigma} \rangle \to \langle \Sigma', [\![\Phi']\!]_{\Sigma'} \rangle$.

is (right-)adjoint.

Proof. This is a simple consequence of the Taut lift theorem. It suffices to prove that given a sS^{op}-initial source $(\langle \Sigma, \Phi \rangle \xrightarrow{\sigma_i} \langle X_i, \Phi_i \rangle)_I$ in $Spa(Sen)^{op}$, $([\![\langle \Sigma, \Phi \rangle]\!]_{\Sigma} \xrightarrow{\sigma_i} [\![\langle X_i, \Phi_i \rangle]\!]_{\Sigma})_I$ is bS-initial. In the light of Proposition 20, this amounts to proving that $[\![\bigcup_{i \in I} Sen(\sigma_i)(\Phi_i)]\!]_{\Sigma} = \bigcap_{i \in I} Bh(\sigma_i)^{-1}([\![\Phi_i]\!]_{\Sigma_i})$. Now:

$$
\begin{array}{lll}
 & b \in [\![\bigcup_{i \in I} Sen(\sigma_i)(\Phi_i)]\!]_{\Sigma} & \\
\text{iff} & b \Vdash^{B}_{\Sigma} \bigcup_{i \in I} Sen(\sigma_i)(\Phi_i) & \\
\text{iff} & b \Vdash^{B}_{\Sigma} Sen(\sigma_i)(\Phi_i) & \text{for all } i \in I \\
\text{iff}^5 & Bh(\sigma_i)(b) \Vdash^{B}_{\Sigma_i} \Phi_i & \text{for all } i \in I \\
\text{iff} & Bh(\sigma_i)(b) \in [\![\Phi_i]\!]_{\Sigma_i} & \text{for all } i \in I \\
\text{iff} & b \in Bh(\sigma_i)^{-1}([\![\Phi_i]\!]_{\Sigma_i}) & \text{for all } i \in I \\
\text{iff} & b \in \bigcap_{i \in I} Bh(\sigma_i)^{-1}([\![\Phi_i]\!]_{\Sigma_i}) &
\end{array}
$$

□

Thus $[\![\_]\!]$ preserves limits: colimits in $Spa(Sen)$ are mapped to limits in $Spa(Bh)$. The functor $[\![\_]\!]$ has an extra useful property: it preserves cartesian morphisms.

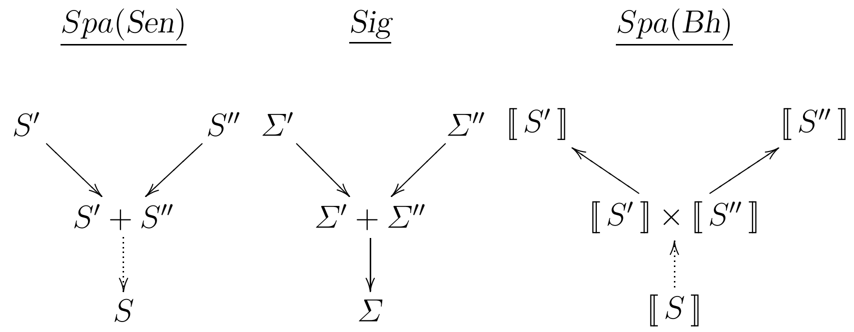
Proposition 25. The functor $[\![\_]\!]$ preserves cartesian morphisms.

Proof. Let $\sigma : \langle \Sigma, Sen(\sigma)(\Phi') \rangle \to \langle \Sigma', \Phi' \rangle$ be a morphism in the category $Spa(Sen)^{op}$. According to Proposition 9, we have that σ is cartesian by sS for $\sigma : \Sigma \to \Sigma'$ in $\langle \Sigma', \Phi' \rangle$. We want to prove that $[\![\sigma]\!] : [\![\langle \Sigma, Sen(\sigma)(\Phi') \rangle]\!] \to [\![\langle \Sigma', \Phi' \rangle]\!]$ is cartesian by bS for σ in $[\![\langle \Sigma', \Phi' \rangle]\!]$. Note that $[\![\sigma]\!] : \langle \Sigma, [\![Sen(\sigma)(\Phi')]\!]_{\Sigma} \rangle \to \langle \Sigma', [\![\Phi']\!]_{\Sigma'} \rangle$. By Proposition 9, we have that $\sigma : \langle \Sigma, Bh(\sigma)^{-1}([\![\Phi']\!]_{\Sigma'}) \rangle \to \langle \Sigma', [\![\Phi']\!]_{\Sigma'} \rangle$ is cartesian by bS for σ in $\langle \Sigma', [\![\Phi']\!]_{\Sigma'} \rangle$, and so it is sufficient to prove that $[\![Sen(\sigma)(\Phi')]\!]_{\Sigma} = Bh(\sigma)^{-1}([\![\Phi']\!]_{\Sigma'})$, which can be done in a similar way to the proof of Proposition 24 □

For the fibre-inclined guy, this can be summarized as saying that $[\![\_]\!]$ is a fibred adjoint.

We can now complete the denotational semantics picture by noting that all operations may be carried at the specification level: the

$[\![\_]\!]$ functor guarantees that they may be transported to the model level.



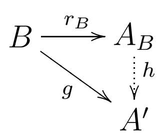
4.4 Theories

We now go a step further and study the relation between theories and interpretation structures. First of all, lets recall what a theory is.

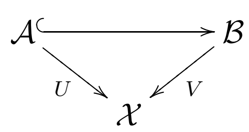
Definition 26. The category of theories is the full concrete subcategory $\langle cSpa(Sen), tS\rangle$ of $\langle Spa(Sen), sS\rangle$ whose objects are closed for the entailment relation $\models$.[6]

We already know some properties of the relation between specifications and interpretation structures in institutions of behaviour. We prove that this relation is maintained if we replace "specifications" by "theories". Again we borrow some results from [1].

Definition 27. A concrete subcategory $\langle \mathcal{A}, U\rangle$ of $\langle \mathcal{B}, V\rangle$ is called *concretely reflective* in $\langle \mathcal{B}, V\rangle$ if for each object B of $\mathcal{B}$ there exists an identity-carried $\mathcal{A}$-reflection arrow $r_B : B \to A_B$.



$V(r_B) = id_{V(B)}$



Proposition 28. A full concrete subcategory of a topological category $\mathcal{X}$ is topological provided it is concretely reflective in $\mathcal{X}$.

[6] As usual, we define that $\Phi \models_\Sigma \varphi$ iff $I \Vdash_\Sigma \Phi$ implies $I \Vdash_\Sigma \varphi$ for each $I \in |Int(\Sigma)|$.

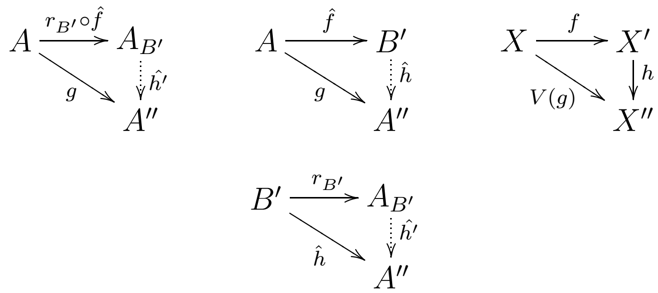
Proposition 29. The concrete category $\langle cSpa(Sen), tS\rangle$ is concretely reflective in $\langle Spa(Sen), sS\rangle$.

Proof. For each $\langle \Sigma, \Phi\rangle \in |Spa(Sen)|$, $id_\Sigma : \langle \Sigma, \Phi\rangle \to \langle \Sigma, \Phi^{\models_\Sigma}\rangle$ is a $cSpa(Sen)$-reflection for $\langle \Sigma, \Phi\rangle$. $\square$

We thus have again the result that $\langle cSpa(Sen), tS\rangle$ is (co)complete iff *Sig* is (co)complete.

Proposition 30. Let $\langle \mathcal{A}, U\rangle$ be concretely reflective in $\langle \mathcal{B}, V\rangle$, and $r_B : B \to A_B$ be an identity-carried $\mathcal{A}$-reflection arrow for each $B \in |\mathcal{B}|$. Then:

- if $\hat{f} : A \to B'$ is cocartesian by V for $f : X \to X'$ on A then $r_{B'} \circ \hat{f} : A \to A_{B'}$ is cocartesian by U for f on A.



- if $\hat{f} : B \to A'$ is cartesian by V for $f : X \to X'$ on A then $\hat{f}' : A_B \to A'$ is cartesian by U for $f : X \to X'$ on A' where $\hat{f}' : A_{B'} \to A''$ is the unique morphism s.t. $\hat{f}' \circ r_B = \hat{f}$.

Corollary 31. The functor tS is a bifibration. In particular, given $\sigma : \Sigma \to \Sigma'$ (and using Proposition 9):

- $\sigma : \langle \Sigma, \Phi\rangle \to \langle \Sigma', Sen(\sigma)(\Phi)^{\models_\Sigma}\rangle$ is cocartesian by tS for σ on $\langle \Sigma, \Phi\rangle$;
- $\sigma : \langle \Sigma, Sen(\sigma)^{-1}(\Phi')\rangle \to \langle \Sigma', \Phi'\rangle$ is cartesian by tS for σ on $\langle \Sigma', \Phi'\rangle$.

Also, the restriction of $[\![\_]\!]$ to $cSpa(Sen)^{op}$ is still an adjunction and preserves cartesian morphisms.

The diagram below relates the several categories and functors presented in the paper.

$$
\begin{array}{ccc}
cSpa(Sen) & \hookrightarrow & Spa(Sen) \\
{\scriptstyle tS}\downarrow & & \downarrow{\scriptstyle sS} \\
Sig & \underset{id}{\longleftrightarrow} & Sig
\end{array}
\qquad
\begin{array}{ccc}
cSpa(Sen)^{op} & \xrightarrow{[\![\_]\!]} & Spa(Bh) \\
{\scriptstyle sS^{op}}\downarrow & & \downarrow{\scriptstyle bS} \\
Sig^{op} & \underset{id}{\longleftrightarrow} & Sig^{op}
\end{array}
$$

5 Concluding remarks

In this paper we addressed a task that has been carried out in the past in a number of situations: the construction of a certain kind of institutions, which we named "behaviour institutions". This kind of institution arises whenever we want to talk about a system in terms of the set of its behaviours and already have a clear understanding and a formalization of individual behaviours.

We noted that, although in the past the construction of the institution has been done in a per-case basis, there is a canonical way of obtaining such institutions.

Furthermore, there is a set of properties shared by institutions of behaviour, regardless of the starting point chosen. In this paper we addressed the particular case of denotational semantics. In particular, the relation between specifications and models in institutions of behaviour was investigated.

This paper casts light on previous work [4, 2, 3, 6–8] by providing a unified and abstract methodology for creating new useful institutions.

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