Combining linear orders with modalities for possible histories

Valentin Goranko* and Alberto Zanardo†

1 Introduction

Two main areas of temporal logics are those of *linear time* and of *branching time*. In the former, time is viewed as a linear sequence of *moments* (see e.g. [5]), while, in the latter, time is pictured in a tree-like fashion: the past of any moment is linearly ordered, but there might be incomparable moments in its future (see e.g. [2, 4]). Linear orders, though, play a crucial role also in logics for branching time. Prior's Ockhamist and Peircean semantical rules for branching time [6], in fact, involve quantification over *histories* in tree-like structures, where histories are maximal linearly ordered sets of moments. Moreover, this quantification can be viewed as (and in Ockhamist logic is) the result of the application of a modal operator (see e.g. [8]). This means that the language and semantics for branching time can be obtained as the combination of languages and semantics for linear time with a modality for possible histories. In this paper, we study various degrees of combining linear time and modal operators and semantics, and we discuss the problem of transferring logical properties from linear to branching time.

1.1 Technical preliminaries

The propositional language for logics of linear time consists of the usual Boolean part plus the temporal operators P and F (with dual operators $H = \neg P \neg$ and $G = \neg F \neg$). The semantics of these operators is the usual Kripke semantics for modal logics: given any linear order $\mathcal{L} = \langle X, < \rangle$ and any evaluation V of the propositional variables in X, truth of Boolean combinations of formulae in the model $M = \langle \mathcal{L}, V \rangle$ is defined in the usual way, while, for tensed formulae, we set

$$\begin{aligned} M, t \models F\theta \quad &\Leftrightarrow \quad M, t' \models \theta\,, \text{ for some } t' > t \\ M, t \models P\theta \quad &\Leftrightarrow \quad M, t' \models \theta\,, \text{ for some } t' < t \end{aligned} \tag{1.1}$$

*Department of Mathematics, Rand Afrikaans University, Johannesburg. E-mail: vfg@rau.ac.za.

†Department of Pure and Applied Mathematics, University of Padova. E-mail azanardo@math.unipd.it.

Given any class C of linear orders, the temporal (P, F) logic of C will be denoted by $L^t(C)$ and the modal logic representing its future fragment by $L^m(C)$.

In the context of this paper, a tree is an irreflexive and transitive order $\mathrm{T} = \langle T, \prec \rangle$ with the *left-linearity* property: if $x \prec y$ and $z \prec y$, then either $x \prec z$, or $z \prec x$, or $x = z$. A *history* in the tree T is a subset of T linearly ordered by $\prec$, which is maximal for inclusion. The set of all histories in the tree T will be written as H(T) and the set of histories passing through t will be written as $\mathrm{H}_t(\mathrm{T})$.

Given any model $M = \langle \mathrm{T}, V \rangle$ based on the tree T, any history h in T and any moment $t \in h$, the truth of a PF-temporal formula θ in M at $\langle h, t \rangle$, denoted $M, h, t \models \theta$, is defined as truth of θ at t in the linear order $\langle h, \prec_{|h} \rangle$, with the evaluation $V' = V_{|h}$.

In a branching time context, the modality $\Diamond$ is read as 'for some possible history' passing through the current moment. Thus, we can set

$$M, h, t \models \Diamond\theta \quad \Leftrightarrow \quad M, h', t \models \theta\,, \text{ for some } h' \in \mathrm{H}_t(\mathrm{T}) \tag{1.2}$$

It is worth observing here that the truth of $\Diamond\theta$ at $\langle t, h \rangle$ in M does not depend on h and hence the right part of (1.2) is often written as $M, t \models \Diamond\theta$. Formulae of this kind are called *state formulae*. Formulae of the form $F\theta$, whose truth is history dependent, are called *path formulae*. As always, we write $\Box$ for the dual operator $\neg\Diamond\neg$.

Given a class of linear orders C we denote by $\mathrm{T}(C)$ the class of trees in which every history belongs to C. We say that $\mathrm{T}(C)$ is the class of trees *generated by* C. Note that $C \subseteq \mathrm{T}(C)$. Moreover, for a set $C = \{\langle X_i, <_i \rangle : i \in I\}$ of linear orders such that $\langle \cup_{i \in I} X_i, \cup_{i \in I} <_i \rangle$ is a tree T, it might happen that $\mathrm{T} \notin T(C)$: T might have *emerging* histories which do not belong to C.

The converse operation is that of passing from a class $\mathcal{C}$ of trees to the class $\mathrm{H}(\mathcal{C}) = \cup_{\mathrm{T} \in \mathcal{C}} \mathrm{H}(\mathrm{T})$. In this way, the expressions H(T) and H({T}) are different names for the same set.

2 A unified approach to branching time semantics

Given a class of linear orders C we can define a hierarchy of modal and temporal logics over the class of trees $\mathrm{T}(C)$ by combining in different degree the temporal operators F, P with the modality $\Diamond$ - see [6], [3], [4], [1], and other works. Here we indicate the natural combinations of modalities and the resulting logic when these are added to propositional logic.

- $\Diamond F$: the ordinary modal logic $L^m_{\mathrm{Prior}}(\mathrm{T}(C))$ on $\mathrm{T}(C)$;
- $\Diamond F, \Diamond P$: the ordinary Priorean temporal logic $L^t_{\mathrm{Prior}}(\mathrm{T}(C))$ on $\mathrm{T}(C)$;
- $\{\Diamond, \Box\} \times \{F\}$: the future fragment of the Peircean logic $L^m_{\mathrm{Peirce}}(\mathrm{T}(C))$ over $\mathrm{T}(C)$;

- $\{\Diamond, \Box\} \times \{F, P\}$: the full Peircean logic $L^t_{\text{Peirce}}(\text{T}(C))$ over $\text{T}(C)$;
- $\{\Diamond, F\}$: the future fragment $L^m_{\text{Ockham}}(\text{T}(C))$ of the Ockhamist logic over $\text{T}(C)$;
- $\{\Diamond, F, P\}$: the full Ockhamist logic ${L^t}_{\text{Ockham}}(\text{T}(C))$ over $\text{T}(C)$.

In Prior and Peircean logics tenses and modalities occur only as part of composed operators. Thus, we will adopt the following notation:

$$\begin{array}{llll} \mathbf{F} := \Box F & \mathbf{f} := \Diamond F & \mathbf{G} := \neg \mathbf{f} \neg & \mathbf{g} := \neg \mathbf{F} \neg \\ \mathbf{P} := \Box P & \mathbf{p} := \Diamond P & \mathbf{H} := \neg \mathbf{p} \neg & \mathbf{h} := \neg \mathbf{P} \neg \end{array}$$

According to (1.1) and (1.2), given any branching time model $M = \langle \text{T}, V \rangle$ and any moment t in T, the truth conditions for $\mathbf{F}$ and $\mathbf{f}$ are

$$\begin{array}{lcl} M, t \models \mathbf{F}\theta & \Leftrightarrow & \forall h \in \text{H}_t \, \exists t' \in h : t \prec t' \text{ and } M, t' \models \theta \\ M, t \models \mathbf{f}\theta & \Leftrightarrow & \exists h \in \text{H}_t \, \exists t' \in h : t \prec t' \text{ and } M, t' \models \theta \end{array} \tag{2.1}$$

The truth conditions for $\mathbf{P}$ and $\mathbf{p}$ can be expressed similarly, by replacing $\prec$ with $\succ$. It must be observed, however, that the left-linearity property of trees implies that the interpretations of $\mathbf{P}$ and $\mathbf{p}$ coincide. These two operators actually agree with the linear time operator P. For this reason, in the sequel we will use only $\mathbf{P}$ and $\mathbf{H}$.

Priorean and Peircean validity (in symbol, $\vDash_{\text{Prior}}$ and $\vDash_{\text{Peirce}}$) are defined on the basis of the above truth conditions, by means of the obvious quantifications over moments and models. On Prior formulae θ, $\vDash_{\text{Prior}}$ and $\vDash_{\text{Peirce}}$ are trivially equivalent.

The Ockhamist logic operators are just the linear time operators P and F, plus the modality $\Diamond$, and their truth conditions are given by (1.1) and (1.2), which define the notion of Ockhamist validity ($\vDash_{\text{Ockham}}$). It must be observed that, in general, the truth of Ockhamist formulae in a model is relative to pairs $\langle t, h \rangle$.

Note that for each of the Peircean and Ockhamist logic, and their future fragments, there are (at least) two different natural semantics: over *bundled trees* ([3], [8]), and over *complete bundled trees* which is equivalent to the one considered above. In the bundled tree semantics, the quantifications over H(T) is replaced by a quantification over an arbitrary subset (bundle) $\mathcal{B}$ of H(T) with the property that $\cup \mathcal{B} = T$.

The Ockhamist formula $\Box G \Diamond F \Box p \to \Diamond G F \Box p$ ([3]) is valid in all complete trees, but can be falsified in some bundled tree. On Peircean validity, instead, the two semantics coincide [4]. This does not mean, though, that any formula which is valid in a specific tree T is also valid in any bundled tree based on T. The formula $\mathbf{G}(p \to \mathbf{f}p) \wedge \mathbf{f}p \to \mathbf{gf}p$, for instance, is Peirce-valid on all complete binary ω-trees[1] but fails on any bundled ω-tree where exactly one history is removed and p is true precisely at all moments of that history.

[1] A binary ω-tree is a tree in which every history is isomorphic to the set ω of natural numbers and every moment has exactly two immediate successors.

3 Translations

The linear time PF-language (or its future fragment) can be embedded into the languages of Priorean, Peircean, and Ockhamist logics. In the case of Priorean logic, the obvious embedding is given by $F \mapsto \mathbf{f}$ and $P \mapsto \mathbf{P}$. Also for Ockhamist logic there is only one possible choice, namely, the identical embedding of the linear time operators.

The situation is more interesting with Peircean logic where we have many possible choices. In principle, each occurrence of F in a linear time formula θ can be translated either to $\mathbf{F}$ or to $\mathbf{f}$ and hence, if θ has n occurrences of F, then there are 2^n possible ways of translating it to a Peircean formula.

In general, arbitrary translations of linear time formulae to Peircean formulae do not preserve validity. The simplest example is the formula

$$Fp \wedge Fq \rightarrow F(p \wedge Fq) \vee F(q \wedge Fp) \vee F(p \wedge q) \tag{3.1}$$

which expresses that time is linear towards the future. Both constant translations, $F \mapsto \mathbf{F}$ and $F \mapsto \mathbf{f}$, transform this formula into non-valid Peircean formulas. On the other hand, linearity towards the future can also be expressed by the formula

$$Fp \rightarrow G(p \vee Fp \vee Pp) \tag{3.2}$$

and the translation $\mathbf{F}p \rightarrow \mathbf{G}(p \vee \mathbf{F}p \vee \mathbf{P}p)$ of this formula is Peirce valid, as well as the translation $\mathbf{f}p \rightarrow \mathbf{g}(p \vee \mathbf{f}p \vee \mathbf{P}p)$.

Thus, a natural question arises to determine the syntactic conditions under which a translation preserves validity, and, in general, the semantical behaviour of translated formulas. Two limit translations can be considered, for instance. The *weakest translation* τ^w (of linear time language to Peircean language) replaces every *positive* occurrence of F by $\mathbf{f}$ and every *negative* occurrence by $\mathbf{F}$. The *strongest translation* τ^s works the other way around.

Every point-wise valuation V (assigning a set of moments to every propositional variable) over a tree T, restricted to any $h \in \mathrm{H(T)}$ determines a valuation V_h on that linear order. Thus, for any Prior formula θ, it makes sense to set, as an auxiliary notion,

$$\mathrm{T}, V, t \vDash_{\mathrm{Peirce}} \Diamond\theta \quad \text{iff} \quad \exists h \in \mathrm{H}_t(\mathrm{T}) : h, V_h, t \vDash_{\mathrm{Prior}} \theta$$

and similarly for $\mathrm{T}, V, t \vDash_{\mathrm{Peirce}} \Box\theta$. On the basis of this definition, we have that, for every linear time formula θ,

$$\vDash_{\mathrm{Peirce}} \Diamond\theta \rightarrow \tau^w(\theta) \quad \text{and} \quad \vDash_{\mathrm{Peirce}} \tau^s(\theta) \rightarrow \Box\theta \tag{3.3}$$

Indeed, note first that after driving all negations in $\tau^w(\theta)$ inside the temporal operators, only operators $\mathbf{f}$ and $\mathbf{g}$ will occur in $\tau^w(\theta)$, i.e. every temporal operator will be prefixed by a $\Diamond$. Then, it suffices to use the validities $\vDash \Diamond\alpha \vee \Diamond\beta \leftrightarrow \Diamond(\alpha \vee \beta)$, $\vDash \Diamond(\alpha \wedge \beta) \rightarrow \Diamond\alpha \wedge \Diamond\beta$, $\vDash \Diamond F\alpha \rightarrow \Diamond F \Diamond\alpha$, and

$\models \Diamond G\alpha \rightarrow \Diamond G\Diamond\alpha$ to pull all $\Diamond' s$ in front of the formula and eventually show that $\models \Diamond\theta \rightarrow \tau^w(\theta)$. Likewise, but dually, for $\tau^s(\theta)$.

A first consequence of (3.3) is that, for any Priorean formula θ, tree T, valuation V on T, and $t \in T$:

$$h, V_h, t \models_{\text{Prior}} \theta \text{ for some } h \in \text{H(T)} \quad \Rightarrow \quad \text{T}, V, t \models_{\text{ Peirce}} \tau^w(\theta) \tag{3.4}$$

and, taking into account that $\neg\tau^w(\theta) \equiv \tau^s(\neg\theta)$,

$$\text{T}, V, t \models_{\text{Peirce}} \tau^s(\theta) \quad \Rightarrow \quad h, V_h, t \models_{\text{Prior}} \theta \text{ for every } h \in \text{H(T)} \tag{3.5}$$

These results imply in turn that, for any Priorean formula θ and any class of linear orders C,

$$C \models_{\text{Prior}} \theta \quad \text{iff} \quad \text{T}(C) \models_{\text{Peirce}} \tau^w(\theta) \tag{3.6}$$

Proof. The implication from right to left is immediate, because $C \subset \text{T}(C)$ and both semantics coincide on linear orders. Now, suppose $\text{T}(C) \not\models_{\text{Peirce}} \tau^w(\theta)$, i.e. $\text{T}, V, t \models_{\text{Peirce}} \neg\tau^w(\theta)$, hence $\text{T}, V, t \models_{\text{Peirce}} \tau^s(\neg\theta)$ for some $\text{T} \in \text{T}(C)$. Then (3.5) implies $h, V_h, t \models_{\text{Prior}} \neg\theta$ for every $h \in \text{H(T)}$, but $\text{H(T)} \subseteq C$. ■

As we see from the proof of (3.6), the claim can be strengthened. In this paper we will establish a syntactic characterization of the translations for which that claim still holds.

In general, the set of translations of a given Prior formula θ can be endowed with a lattice structure by letting $\tau \leq \tau'$ whenever $\tau(\theta)$ can be obtained from $\tau'(\theta)$ by replacing some (possibly no) positive occurrence of $\mathbf{f}$ by $\mathbf{F}$ and some (possibly no) negative occurrence of $\mathbf{F}$ by $\mathbf{f}$. In this way τ^w and τ^s turn out to be the top and the bottom or the lattice, respectively. On the basis of the Peirce validity $\mathbf{F}\alpha \rightarrow \mathbf{f}\alpha$ it can be proved that

$$\tau \leq \tau' \quad \Rightarrow \quad \models_{\text{Peirce}} \tau(\theta) \rightarrow \tau'(\theta)$$

4 Transfer of properties

In this paper we are investigating transfer of logical properties, such as definability, axiomatizations and decidability between various linear time logics $L^t(C)$ (or $L^m(C)$) and their branching time counterparts $L^t(\text{T}(C))$ with Priorean, or Peircean, or Ockhamist semantics.

4.1 Transfer of definability

Which modally definable properties of linear orders in a given class C transfers to properties of all histories in $\text{T}(C)$? More specifically, given a property of linear orders and a formula θ which defines that property in the class C, we wonder whether some translation of θ defines the same property for all histories in $\text{T}(C)$.

The answer is trivial when Ockhamist logic is considered because Ockhamist tense operators are linear time operators and hence any definable property is transferred to a property which is definable in tree by means of the same formula.

Passing from Peircean definability to linear time definability is trivial as well. If the Peircean formula ϕ defines a given property of histories in some $\mathrm{T}(C)$, then, for any translation τ, $\tau^{-1}(\phi)$ defines that property in C. In fact, the elements of C are particular trees in $\mathrm{T}(C)$ and, on linear orders, the interpretations of both $\mathbf{F}$ and $\mathbf{f}$ coincide with the interpretation of F.

General transferability results from linear time logics to Peircean logics have not been established yet. As a further consequence of (3.3), we have that, if θ defines the property $\mathcal{P}$ in the class of linear orders, then, for every tree T:

(1) $\mathrm{T}, t \vDash_{\mathrm{Peirce}} \tau^w(\theta)$ whenever a history passing through t has $\mathcal{P}$,

and

(2) $\mathrm{T}, t \vDash_{\mathrm{Peirce}} \tau^s(\theta)$ implies that every history passing through t has $\mathcal{P}$.

Here are some examples of definable properties of linear orders which are also definable in trees by Peircean formulas.

Density	$\mathbf{f}p \to \mathbf{ff}p$ (or $\mathbf{F}p \to \mathbf{FF}p$)
Discreteness	$(\mathbf{f}\top \to ((p \wedge \mathbf{H}p) \to \mathbf{FH}p)) \wedge$ $(\mathbf{P}\top \to ((p \wedge \mathbf{g}p)) \to \mathbf{Pg}p)$
Dedekind Continuity	$\mathbf{FG}\neg p \wedge \mathbf{F}p \to \mathbf{F}(\mathbf{g}\neg p \wedge \mathbf{Hf}p)$
Isomorphism to $\mathbb{Z}$	$\mathbf{H}p \to \mathbf{P}p \wedge \mathbf{G}p \to \mathbf{F}p \wedge$ $\mathbf{H}(\mathbf{H}p \to p) \to (\mathbf{PH}p \to \mathbf{H}p) \wedge$ $\mathbf{G}(\mathbf{G}p \to p) \to (\mathbf{FG}p \to \mathbf{G}p)$

4.2 Transfer of satisfiability/validity, axiomatizations, decidability

How do satisfiability and validity transfer between linear time logics and their branching time counterparts? Accordingly, when are axiomatizations and decidability results preserved in passing between these?

With Ockhamist logic, for instance, the translation τ is the identical embedding and Ockhamist PF-validity is linear time validity. Thus, any axiomatization of $L^t(C)$ is also an axiomatization of the PF fragment of $L^t_{\mathrm{Ockham}}(\mathrm{T}(C))$. Decidability transfers similarly.

The situation is quite different, though, if we adopt a different definition of $\mathrm{T}(C)$, and we let this class be the set of bundled trees $\langle \mathrm{T}, \mathcal{B} \rangle$ in which $\mathcal{B} \subseteq C$. This new perspective is actually suggested by the *Kamp frames* considered in [7]. With this new definition of $\mathrm{T}(C)$, emerging histories might appear and, at this stage, no proof of the preservation results considered above seems to be available.

5 Concluding remarks

This paper aims at systematic investigation of the logical aspects and virtues of combining linear orders as semantics for modal and temporal logics, with modalities for possible histories, resulting into a variety of branching time logics.

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