

Why are combined modal logics so robustly undecidable?

Frank Wolter

Department of Computer Science, University of Liverpool,
Liverpool L69 7ZF, U.K.

One of the main reasons for the success of modal logics in computer science is their unusual robust decidability. Indeed, standard modal logics like polymodal **K**, **S4**, and **S5**, temporal logics like **LTL** and **CTL**, dynamic logics like **PDL**, epistemic logics like **S5** with common knowledge, description logics like **ALC** and **SHIQ**, and modal spatial logics like **S4_u**, are decidable in EXPTIME [7, 4].

However, modern applications of modal languages often require rather complex formal models and corresponding languages that are capable of reflecting different features of an application domain. It is not sufficient to work with just one sort of modal operators (say, epistemic or temporal) but subtle combinations of different families of modal operators are required. For example,

- to analyze the behaviour of a multi-agent distributed system we may need a formalism containing both epistemic operators for capturing knowledge of agents and temporal operators for taking care of the development of this knowledge in time. In other words, we should construct a suitable combination of epistemic and temporal logics [2].
- To describe the behaviour of spatial objects (regions) changing over time we may need combinations of spatial modal logics with temporal logics;
- to analyze the dynamics of ontologies and the knowledge of agents about ontologies we may need combinations of description logics (as the underlying language for ontologies) with dynamic modal logics and epistemic logics.
- Fragments of first-order logic may be required to describe an application domain. In this case, instead of propositional modal logics, first-order modal logics are required; i.e., we have to combine (fragments) of first-order logic with propositional modal logics.

While decidability is preserved under forming combinations of modal logics without interaction axioms or constraints (i.e., fusions) [8, 1] this situation changes drastically as soon as some kind of interaction between the modalities is required. In contrast to standard modal logics, combined modal logics often exhibit rather nasty computational properties, and the standard toolkit of modal logic, e.g.,

- proving the finite model property by some kind of filtration;
- proving a variant of the tree-model property and applying automata-based decision procedures [7, 4];
- providing an embedding into a decidable fragment of first-order logic [4]

is no longer directly available. In fact, straightforward constructions of combined modal logics from simple one-dimensional ones will almost certainly result in computationally useless ‘monsters’.

The aim of this presentation is twofold: (i) to discuss possible explanations for this phenomenon - *the robust undecidability of combined modal logics* - and (ii) to use the insights gained from this discussion in order to provide methodologies for constructing computationally well-behaved combined modal logics.

We will argue that the high computational complexity or even undecidability of most combined modal logics is explained by the two- or many-dimensional structure of their intended models [3]:

- The fact that almost all *three-dimensional* modal logics are undecidable can be intuitively explained by the undecidability of the product $\mathbf{S5}^3$ and its relation to the undecidable 3-variable fragment of first-order logic.
- *Two-dimensional* modal logics are usually at least NEXPTIME-hard. This can be intuitively explained by the NEXPTIME-completeness of the product $\mathbf{S5}^2$ and its relation to the 2-variable fragment of first-order logic. However, the computational properties of two-dimensional modal logics depend - in a way not yet completely understood - on the geometry of the models of the components. We will present partial results for two-dimensional modal logics as well as open problems based on [3].
- Products of $\mathbf{S5}$ with \mathbf{CTL}^* and \mathbf{CTL} may be regarded as $2\frac{1}{2}$ -dimensional modal logics and show again the computational significance of the move from two to three dimensions: the product with \mathbf{CTL}^* is undecidable while the product with \mathbf{CTL} (or \mathbf{LTL}) is decidable [5].
- The decidability/undecidability of fragments of first-order modal logics is closely related to the number of free variables allowed within formulas starting with a modal operator; i.e., formulas of the form $\Box\varphi(\vec{x})$. While *monodic* fragments - which allows for just one free such variable - often exhibits ‘good’ computational properties, two variables usually lead to undecidable fragments already [3]. We will give an overview of explanations for this and discuss open problems.
- Finally, we discuss the method of E-connections to produce decidable and expressive combinations of modal logics of many dimensions [6].

Acknowledgements

This presentation is based on joint work with Ian Hodkinson, Agnes Kurucz, Carsten Lutz and Michael Zakharyashev.

References

- [1] F. Baader, S. Ghilardi, and C. Tinelli. A new combination procedure for the word problem that generalizes fusion decidability results in modal logics. In D. Basin and M. Rusinowitch, editors, *Proceedings of the 2nd International Joint Conference on Automated Reasoning (IJCAR'04)*, Lecture Notes in Artificial Intelligence. Springer-Verlag, 2004. To appear.
- [2] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.
- [3] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyashev. *Many-dimensional modal logics: theory and applications*. Elsevier, 2003.

- [4] E. Grädel. Why are modal logics so robustly decidable? In G. Paun, G. Rozenberg, and A. Salomaa, editors, *Current Trends in Theoretical Computer Science. Entering the 21st Century*, pages 393–408. World Scientific, 2001.
- [5] I. Hodkinson, F. Wolter, and M. Zakharyashev. Decidable and undecidable fragments of first-order branching temporal logics. In *Proceedings of the 17th Annual IEEE Symposium on Logic in Computer Science (LICS'02)*, pages 393–402. IEEE, 2002.
- [6] O. Kutz, C.Lutz, F. Wolter, and M. Zakharyashev. E-connections of abstract description systems. *Artificial Intelligence*, 156:1 – 73, 2004.
- [7] M. Vardi. Why is modal logic so robustly decidable? In *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, volume 31, pages 149–184. AMS, 1997.
- [8] F. Wolter. Fusions of modal logics revisited. In M. Kracht, M. De Rijke, H. Wansing, and M. Zakharyashev, editors, *Advances in Modal Logic*, volume 1, pages 361–379. CSLI, Stanford, 1997.