

# Analysis of two fragments with negation and without implication of the logic of residuated lattices

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## Abstract

The logic of (commutative integral bounded) residuated lattices is known under different names in the literature: monoidal logic [9], intuitionistic logic without contraction [1],  $H_{BCK}$  [13], etc. It is usually given, up to definitional equivalence, in the language  $\langle \vee, \wedge, *, \neg, \rightarrow, 0, 1 \rangle$ . It was shown in [1] that this logic, denoted there by  $IPC^* \setminus c$ , is algebraizable (the variety of residuated lattices is its equivalent algebraic semantics) and that is the external deductive system associated to the Gentzen system determined by the sequent calculus  $\mathbf{FL}_{ew}$  [10, 12]. This logic is decidable [5]. The  $\langle \rightarrow \rangle$ -fragment and the  $\langle *, \rightarrow \rangle$ -fragment of this logic (i.e., the BCK-logic and the POCRIM-logic) are also algebraizable and decidable [5].

This paper contains a summary of the results obtained in [6] about the  $\langle \vee, *, \neg, 0, 1 \rangle$ -fragment and the  $\langle \vee, \wedge, *, \neg, 0, 1 \rangle$ -fragment of the logic of residuated lattices.

As regards the algebraic aspects of this study, we introduce the notion of pseudocomplementation with respect to the monoidal operation  $*$  (see [3] in the case of  $\wedge$ ). We then define two new classes of algebras: the class of commutative integral bounded semilatticed pseudocomplemented monoids, denoted by  $\mathbf{CIBPM}^{s\ell}$ , and the class of commutative integral bounded latticed pseudocomplemented monoids, denoted by  $\mathbf{CIBPM}^\ell$ . We show that these classes of algebras are varieties whose quasiequational theories are decidable. Their members are exactly the subreducts of the variety of residuated lattices, i.e., every  $\mathbf{CIBPM}^{s\ell}$ -algebra and every  $\mathbf{CIBPM}^\ell$ -algebra is embeddable into a residuated lattice.

As regards the logical aspects of this study, we introduce two sequent calculi:  $\mathbf{FL}_{ew}[V, *, \neg]$  and  $\mathbf{FL}_{ew}[V, \wedge, *, \neg]$ . They are obtained from the well known contraction-free calculus  $\mathbf{FL}_{ew}$  (in the language  $\langle \vee, \wedge, *, \neg, \rightarrow, 0, 1 \rangle$ ) [12]. The former,  $\mathbf{FL}_{ew}[V, *, \neg]$ , is obtained by deleting the rules of additive conjunction  $\wedge$  and implication  $\rightarrow$  from  $\mathbf{FL}_{ew}$ . And  $\mathbf{FL}_{ew}[V, \wedge, *, \neg]$  results from deleting the rules of implication  $\rightarrow$  from  $\mathbf{FL}_{ew}$ . It can be shown that  $\mathbf{CIBPM}^{s\ell}$  ( $\mathbf{CIBPM}^\ell$ ) is the equivalent variety semantics [14, 8] of the intuitionistic Gentzen system associated to the sequent calculi  $\mathbf{FL}_{ew}[V, *, \neg]$  ( $\mathbf{FL}_{ew}[V, \wedge, *, \neg]$ ). As a consequence, we have that the variety  $\mathbf{CIBPM}^{s\ell}$  ( $\mathbf{CIBPM}^\ell$ ) is an algebraic semantics, with defining equation  $p \approx 1$ , for the external deductive system  $\mathcal{S}_e[V, *, \neg]$  ( $\mathcal{S}_e[V, \wedge, *, \neg]$ ) associated to  $\mathbf{FL}_{ew}[V, *, \neg]$  ( $\mathbf{FL}_{ew}[V, \wedge, *, \neg]$ ).

Moreover, we show a generalization of [11, Corollary 9]:  $\mathcal{S}_e[\vee, *, \neg]$  ( $\mathcal{S}_e[\vee, \wedge, *, \neg]$ ) is the  $\langle \vee, *, \neg, 0, 1 \rangle$ -fragment ( $\langle \vee, \wedge, *, \neg, 0, 1 \rangle$ -fragment) of the logic of residuated lattices. We show that  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are decidable. We also show that  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are not protoalgebraic [4, 7], so they are not algebraizable and also there is no definable implication connective for them.

## 1 Preliminary concepts

In this section we recall some concepts and results about Gentzen systems and the algebraization of these systems which will be used in this paper (for more information see [14], [15], [8]).

### 1.1 Gentzen systems

Let  $\mathcal{L}$  be a propositional language. We will denote by  $Fm_{\mathcal{L}}$  the set of  $\mathcal{L}$ -formulas and by  $\mathcal{F}m_{\mathcal{L}}$  the algebra of  $\mathcal{L}$ -formulas. Let  $\alpha$  and  $\beta$  be some subsets of the set  $\omega$  of natural numbers. A  $\mathcal{L}$ -sequent of type  $(\alpha, \beta)$  is a pair  $(\Gamma, \Delta)$  of finite sequences of  $\mathcal{L}$ -formulas such that the length of  $\Gamma$  belongs to  $\alpha$  and the length of  $\Delta$  belongs to  $\beta$ , that is,  $(\Gamma, \Delta) \in Fm_{\mathcal{L}}^m \times Fm_{\mathcal{L}}^n$  for some  $m \in \alpha$  and  $n \in \beta$ . We will write  $\Gamma \Rightarrow \Delta$  instead of  $(\Gamma, \Delta)$ . We will denote by  $Seq_{\mathcal{L}}^{(\alpha, \beta)}$  the set of  $\mathcal{L}$ -sequents of type  $(\alpha, \beta)$ .

A consequence relation  $\vdash_{\mathcal{G}}$  on the set  $Seq_{\mathcal{L}}^{(\alpha, \beta)}$  is said to be *invariant under substitutions* if, for every  $h \in Hom(Fm_{\mathcal{L}}, Fm_{\mathcal{L}})$ ,

$$\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta \text{ implies } \{h(\Gamma_i \Rightarrow \Delta_i) : i \in I\} \vdash_{\mathcal{G}} h(\Gamma \Rightarrow \Delta),$$

where  $h(\varphi_0, \dots, \varphi_{m-1} \Rightarrow \psi_0, \dots, \psi_{n-1})$  stands for  $h\varphi_0, \dots, h\varphi_{m-1} \Rightarrow h\psi_0, \dots, h\psi_{n-1}$ .

A GENTZEN SYSTEM OF TYPE  $(\alpha, \beta)$  is a pair  $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$  where  $\vdash_{\mathcal{G}}$  is a relation which is finitary and invariant under substitutions on the set  $Seq_{\mathcal{L}}^{(\alpha, \beta)}$ . If  $T \cup \{\Gamma \Rightarrow \Delta, \Pi \Rightarrow \Lambda\} \subseteq Seq_{\mathcal{L}}^{(\alpha, \beta)}$ , we will write  $T, \Gamma \Rightarrow \Delta \vdash_{\mathcal{G}} \Pi \Rightarrow \Lambda$  for  $T \cup \{\Gamma \Rightarrow \Delta\} \vdash_{\mathcal{G}} \Pi \Rightarrow \Lambda$ .

We will say that two sequents  $\Gamma \Rightarrow \Delta$  and  $\Pi \Rightarrow \Lambda$  are  $\mathcal{G}$ -equivalent and we will write  $\Gamma \Rightarrow \Delta \dashv\vdash_{\mathcal{G}} \Pi \Rightarrow \Lambda$  if it holds that  $\Gamma \Rightarrow \Delta \vdash_{\mathcal{G}} \Pi \Rightarrow \Lambda$  and  $\Pi \Rightarrow \Lambda \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta$ . A sequent  $\Gamma \Rightarrow \Delta \in Seq_{\mathcal{L}}^{(\alpha, \beta)}$  is  $\mathcal{G}$ -derivable if  $\emptyset \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta$ .

Clearly, a deductive system  $\mathcal{S}$  can be seen as a Gentzen system  $\mathcal{G}_{\mathcal{S}}$  of type  $(\{0\}, \{1\})$ , if we identify a formula  $\varphi$  with the sequent  $\emptyset \Rightarrow \varphi$ . Furthermore, if  $K$  is a class of algebras such that the equational consequence relation determined by  $K$  is finitary, then  $S_K = \langle \mathcal{L}, \models_K \rangle$  will be considered as a Gentzen system  $\mathcal{G}_K$  of type  $(\{1\}, \{1\})$ , if we identify an equation  $\varphi \approx \psi$  with the sequent  $\varphi \Rightarrow \psi$ .

### 1.2 Fragments

Let  $\mathcal{G}$  be a Gentzen system of type  $(\alpha, \beta)$  and let  $\mathcal{L}'$  be a sublanguage of  $\mathcal{L}$ . The  $\mathcal{L}'$ -fragment of  $\mathcal{G}$  is the Gentzen system  $\mathcal{G}' = \langle \mathcal{L}', \vdash_{\mathcal{G}'} \rangle$  of type  $(\alpha, \beta)$  defined by:

$$T \vdash_{\mathcal{G}'} \Gamma \Rightarrow \Delta \text{ iff } T \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta, \text{ for all } T \cup \{\Gamma \Rightarrow \Delta\} \subseteq Seq_{\mathcal{L}'}^{(\alpha, \beta)}.$$

Let  $\alpha' \subseteq \alpha$  and  $\beta' \subseteq \beta$ . The  $(\alpha', \beta')$ -fragment of  $\mathcal{G}$  is the Gentzen system  $\mathcal{G}' = \langle \mathcal{L}, \vdash_{\mathcal{G}'} \rangle$  of type  $(\alpha', \beta')$  defined by:

$$T \vdash_{\mathcal{G}'} \Gamma \Rightarrow \Delta \text{ iff } T \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta, \text{ for all } T \cup \{\Gamma \Rightarrow \Delta\} \subseteq Seq_{\mathcal{L}}^{(\alpha', \beta')}.$$

### 1.3 Sequent calculus

An  $(\mathcal{L}, (\alpha, \beta))$ -sequent calculus is a set of  $(\mathcal{L}, (\alpha, \beta))$ -rules. Every  $(\mathcal{L}, (\alpha, \beta))$ -sequent calculus  $LX$  determines a Gentzen system  $\mathcal{G}_{LX} = \langle \mathcal{L}, \vdash_{LX} \rangle$  of type  $(\alpha, \beta)$  in the following way:

Let  $T \cup \{\Gamma \Rightarrow \Delta\} \subseteq \text{Seq}_{\mathcal{L}}^{(\alpha, \beta)}$ . We will say that  $\Gamma \Rightarrow \Delta$  is *deduced* from  $T$  in the Gentzen system  $\mathcal{G}_{LX}$  and we will write  $T \vdash_{LX} \Gamma \Rightarrow \Delta$  iff there is a finite sequence of  $\mathcal{L}$ -sequents  $\Gamma_0 \Rightarrow \Delta_0, \dots, \Gamma_{n-1} \Rightarrow \Delta_{n-1}$  (which is called a *proof* of  $\Gamma \Rightarrow \Delta$  from  $T$ ) such that  $\Gamma_{n-1} \Rightarrow \Delta_{n-1} = \Gamma \Rightarrow \Delta$  and for each  $i < n$  one of the following conditions holds:

- (i)  $\Gamma_i \Rightarrow \Delta_i$  is an instance of an axiom of  $LX$ ;
- (ii)  $\Gamma_i \Rightarrow \Delta_i \in T$ ;
- (iii)  $\Gamma_i \Rightarrow \Delta_i$  is obtained from  $\{\Gamma_j \Rightarrow \Delta_j : j < i\}$  by using a rule  $r$  of  $LX$ , i. e.,  $\frac{T}{\Gamma_i \Rightarrow \Delta_i} \in r$  for some  $T \subseteq \{\Gamma_j \Rightarrow \Delta_j : j < i\}$ .

In this case we will say that  $\mathcal{G}_{LX}$  is the *Gentzen system determined by the sequent calculus  $LX$* . Note that we use the rules of the calculus to obtain sequents from sets of sequents (and so not only from the empty set).

### 1.4 Equivalence of Gentzen systems

From now on let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be Gentzen systems over the same language  $\mathcal{L}$  of type  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  respectively. A  $((\alpha_1, \beta_1), (\alpha_2, \beta_2))$ -translation is a set

$$\tau = \{\tau_{(m,n)} : m \in \alpha_1, n \in \beta_1\},$$

where every  $\tau_{(m,n)}$  is a finite set of sequents  $\text{Seq}_{\mathcal{L}}^{(\alpha_2, \beta_2)}$  in  $m + n$  variables  $p_0, \dots, p_{m-1}, q_0, \dots, q_{n-1}$  (at most) if  $(m, n) \neq (0, 0)$ , and in one variable  $p_0$  (at most) if  $(m, n) = (0, 0)$ .

Given  $\Gamma \Rightarrow \Delta = (\varphi_0, \dots, \varphi_{m-1}) \Rightarrow (\psi_0, \dots, \psi_{n-1}) \in \text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ , we will write  $\tau(\Gamma \Rightarrow \Delta) = \tau_{(m,n)}(\Gamma \Rightarrow \Delta)$ , the result of replacing the variable  $p_i$  by  $\varphi_i$ ,  $i < m$ , and the variable  $q_j$  by  $\psi_j$ ,  $j < n$ , in every sequent of  $\tau_{(m,n)}$ . If  $\emptyset \Rightarrow \emptyset \in \text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ , we will write  $\tau(\emptyset \Rightarrow \emptyset) = \tau_{(0,0)}(p_0)$ . If  $T \subseteq \text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ ,  $\tau(T)$  will denote the set  $\cup \{\tau(\Gamma \Rightarrow \Delta) : \Gamma \Rightarrow \Delta \in T\}$ .

We say that  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are *equivalent* if there is a translation  $\tau$  of the set of sequents  $\text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$  in the set of sequents  $\text{Seq}_{\mathcal{L}}^{(\alpha_2, \beta_2)}$  and a translation  $\rho$  of the set of sequents  $\text{Seq}_{\mathcal{L}}^{(\alpha_2, \beta_2)}$  in the set of sequents  $\text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ , such that

- (i)  $T \vdash_{\mathcal{G}_1} \Gamma \Rightarrow \Delta$  iff  $\tau(T) \vdash_{\mathcal{G}_2} \tau(\Gamma \Rightarrow \Delta)$  for all  $T \cup \{\Gamma \Rightarrow \Delta\} \subseteq \text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ .
- (ii)  $T \vdash_{\mathcal{G}_2} \Gamma \Rightarrow \Delta$  iff  $\rho(T) \vdash_{\mathcal{G}_1} \rho(\Gamma \Rightarrow \Delta)$  for all  $T \cup \{\Gamma \Rightarrow \Delta\} \subseteq \text{Seq}_{\mathcal{L}}^{(\alpha_2, \beta_2)}$ .
- (iii)  $\Gamma \Rightarrow \Delta \dashv\vdash_{\mathcal{G}_2} \tau\rho(\Gamma \Rightarrow \Delta)$  for all  $\Gamma \Rightarrow \Delta \in \text{Seq}_{\mathcal{L}}^{(\alpha_2, \beta_2)}$ .
- (iv)  $\Gamma \Rightarrow \Delta \dashv\vdash_{\mathcal{G}_1} \rho\tau(\Gamma \Rightarrow \Delta)$  for all  $\Gamma \Rightarrow \Delta \in \text{Seq}_{\mathcal{L}}^{(\alpha_1, \beta_1)}$ .

In fact this definition is redundant because (i) and (iii) are equivalent to (ii) and (iv) (see [15, Proposition 2.1]).

## 1.5 Algebraization of Gentzen systems

Let  $K$  be a class of algebras. We will denote by  $\mathcal{S}_K$  the equational system associated with  $K$ . Gentzen system  $\mathcal{G}$  is said to be ALGEBRAIZABLE WITH EQUIVALENT ALGEBRAIC SEMANTICS  $K$  if  $\mathcal{G}$  and  $\mathcal{S}_K$  are equivalent as Gentzen systems.

If  $K$  is an equivalent algebraic semantics for a Gentzen system  $\mathcal{G}$ , then so is the quasivariety  $K^Q$  generated by  $K$  [15, Corollary 2.2]. Moreover, if  $K$  and  $K'$  are equivalent algebraic semantics for  $\mathcal{G}$ , then  $K$  and  $K'$  generates the same quasivariety [15, Corollary 2.4]. This quasivariety is called THE EQUIVALENT QUASIVARIETY SEMANTICS for  $\mathcal{G}$ .

## 2 The calculi $\mathbf{FL}_{\text{ewc}}$ , $\mathbf{FL}_{\text{ew}}$ , $\mathbf{FL}_{\text{ew}}[\vee, *, \neg]$ , $\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]$

In this section we recall the definition of the sequent calculi (cut free)  $\mathbf{FL}_{\text{ewc}}$  and  $\mathbf{FL}_{\text{ew}}$  (Cf. [10], [12]) in the language  $\mathcal{L} = \{\vee, \wedge, *, \rightarrow, \neg, 0, 1\}$  and we define the calculi  $\mathbf{FL}_{\text{ew}}[\vee, *, \neg]$  and  $\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]$  obtained by restriction of the calculus  $\mathbf{FL}_{\text{ew}}$  to the languages  $\{\vee, *, \neg, 0, 1\}$  and  $\{\vee, \wedge, *, \neg, 0, 1\}$  respectively.

**Definition 2.1.** Let  $\mathcal{L} = \{\vee, \wedge, *, \rightarrow, \neg, 0, 1\}$  be a propositional language of type  $(2, 2, 2, 2, 1, 0, 0)$ . Let  $\varphi, \psi$  be  $\mathcal{L}$ -formulas;  $\Gamma, \Pi, \Sigma$  finite sequences (possibly empty) of  $\mathcal{L}$ -formulas and  $\Delta$  a sequence of at most one formula.  $\mathbf{FL}_{\text{ewc}}$  is the calculus of  $\mathcal{L}$ -sequents of type  $(\omega, \{0, 1\})$  defined by the following axioms and rules:

*Axioms:*

$$\varphi \Rightarrow \varphi \quad (\text{Axiom 1}) \quad 0 \Rightarrow \emptyset \quad (\text{Axiom 2}) \quad \emptyset \Rightarrow 1 \quad (\text{Axiom 3})$$

*Structural rules:*

*Cut:*

$$\frac{\Gamma \Rightarrow \varphi \quad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta} \quad (\text{Cut})$$

*Exchange:*

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \quad (e \Rightarrow)$$

*Weakening:*

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi} \quad (\Rightarrow w)$$

*Contraction:*

$$\frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (c \Rightarrow)$$

*Rules of introduction of connectives:*

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta \quad \Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \vee \psi, \Gamma \Rightarrow \Delta} \quad (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \quad (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \quad (\Rightarrow \vee_2)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad (\wedge_1 \Rightarrow) \quad \frac{\Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad (\wedge_2 \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \quad (\Rightarrow \wedge)$$

$$\begin{array}{ccc}
\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta} \quad (* \Rightarrow) & & \frac{\Gamma \Rightarrow \varphi \quad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi} \quad (\Rightarrow *) \\
\frac{\Gamma \Rightarrow \varphi \quad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta} \quad (\rightarrow \Rightarrow) & & \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \quad (\Rightarrow \rightarrow) \\
\frac{\Gamma \Rightarrow \varphi}{\neg \varphi, \Gamma \Rightarrow \emptyset} \quad (\neg \Rightarrow) & & \frac{\varphi, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \neg \varphi} \quad (\Rightarrow \neg)
\end{array}$$

**Definition 2.2.**  $\mathbf{FL}_{\text{ew}}$  is the calculus of  $\mathcal{L}$ -sequents of type  $(\omega, \{0, 1\})$  obtained from  $\mathbf{FL}_{\text{ewc}}$  by deleting the rule  $(c \Rightarrow)$ .

**Theorem 2.3** ([10, Theorem 6]). *Cut elimination holds for  $\mathbf{FL}_{\text{ewc}}$  and  $\mathbf{FL}_{\text{ew}}$ .*

**Definition 2.4.** *The calculus obtained by deleting from  $\mathbf{FL}_{\text{ew}}$  the rules of introduction of the additive conjunction and the implication will be denoted by  $\mathbf{FL}_{\text{ew}}[\vee, *, \neg]^1$ , and the calculus obtained by deleting from  $\mathbf{FL}_{\text{ew}}$  the rules for the implication will be denoted by  $\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]$ . The Gentzen systems associated to the calculi  $\mathbf{FL}_{\text{ew}}[\vee, *, \neg]$  and  $\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]$  will be denoted by  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$ , respectively.*

**Theorem 2.5.**  $\mathbf{FL}_{\text{ew}}[\vee, *, \neg]$  and  $\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]$  satisfy the cut elimination theorem.

### 3 Equivalence between $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ , the deductive system $\widehat{IPC}^* \setminus c$ and the variety of residuated lattices

In this section we will define the deductive system  $\widehat{IPC}^* \setminus c$ . It is easy to see that this system is definitionally equivalent to  $IPC^* \setminus c$  [1],  $H_{BCK}$  [13] and  $ML$  [9]. We will also state the algebraization of the Gentzen system  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ .

**Definition 3.1.**  $\widehat{IPC}^* \setminus c$  is the deductive system in the language  $\mathcal{L} = \{\vee, \wedge, *, \rightarrow, \neg, 0, 1\}$  of type  $(2, 2, 2, 2, 1, 0, 0)$ , defined by the Modus Ponens rule and the following axioms:

- (A1)  $\varphi \rightarrow 1$
- (A2)  $(\varphi \rightarrow \psi) \rightarrow ((\gamma \rightarrow \varphi) \rightarrow (\gamma \rightarrow \psi))$  (B)
- (A3)  $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \gamma))$  (C)
- (A4)  $\varphi \rightarrow (\psi \rightarrow \varphi)$  (K)
- (A5)  $(\varphi \wedge \psi) \rightarrow \varphi$
- (A6)  $(\varphi \wedge \psi) \rightarrow \psi$
- (A7)  $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
- (A8)  $((\gamma \rightarrow \varphi) \wedge (\gamma \rightarrow \psi)) \rightarrow (\gamma \rightarrow (\varphi \wedge \psi))$
- (A9)  $\psi \rightarrow (\varphi \vee \psi)$

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<sup>1</sup>It would be more accurate to talk about  $\mathbf{FL}_{\text{ew}}[\vee, *, \neg, 0, 1]$  because this is the language where this calculus is given, but for the sake of simplicity we will not do so.

- (A10)  $\varphi \rightarrow (\varphi \vee \psi)$   
(A11)  $(\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\varphi \vee \psi) \rightarrow \gamma))$   
(A12)  $\varphi \rightarrow (\psi \rightarrow (\varphi * \psi))$   
(A13)  $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi * \psi) \rightarrow \gamma)$   
(A14)  $0 \rightarrow \varphi$   
(A15)  $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$   
(A16)  $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$   
(A17)  $\varphi \rightarrow \neg\neg\varphi$

Let us recall the definition of residuated lattice:

**Definition 3.2.** *The class  $\mathbb{RL}$  of COMMUTATIVE INTEGRAL BOUNDED RESIDUATED LATTICE, or RESIDUATED LATTICE for short, is the class of algebras  $\mathcal{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle$  of type  $(2, 2, 2, 2, 1, 0, 0)$  satisfying the following conditions:*

1.  $\langle A, \vee, \wedge, 0, 1 \rangle$  is a bounded lattice with minimum element 0 and maximum element 1,
2.  $\langle A, *, 1 \rangle$  is a commutative monoid,
3.  $\forall a, b \in A, a \rightarrow b = \max \{c \in A : a * c \leq b\}$  (i.e.  $x * z \leq y \Leftrightarrow z \leq x \rightarrow y$ )
4.  $\forall a \in A, \neg a = \neg a \rightarrow 0$ .

It is well known that  $\mathbb{RL}$  is a variety. It was proved in [1, Theorem 22] that  $\mathcal{G}_{LJ^* \setminus c}$  and  $IPC^* \setminus c$  are equivalent and it was also proved in [1, Theorem 22] that the Gentzen system  $\mathcal{G}_{LJ^* \setminus c}$  is algebraizable with equivalent algebraic semantics the variety of residuated lattices. As  $\mathcal{G}_{\mathbf{FL}_{ew}}$  and  $\widehat{IPC}^* \setminus c$  are essentially equivalent to  $\mathcal{G}_{LJ^* \setminus c}$  and  $IPC^* \setminus c$ , respectively, we have the following results:

**Theorem 3.3.**  *$\mathcal{G}_{\mathbf{FL}_{ew}}$  and  $\widehat{IPC}^* \setminus c$  are equivalent, with the translations  $\tau$  from  $\mathcal{G}_{\mathbf{FL}_{ew}}$  to  $\widehat{IPC}^* \setminus c$  and  $\rho$  from  $\widehat{IPC}^* \setminus c$  to  $\mathcal{G}_{\mathbf{FL}_{ew}}$  defined as follows:*

$$\tau_{(m,1)}(p_0, \dots, p_{m-1} \Rightarrow q_0) = \begin{cases} p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{m-1} \rightarrow q_0) \dots)), & \text{if } m \geq 1 \\ q_0, & \text{if } m = 0 \end{cases}$$

$$\tau_{(m,0)}(p_0, \dots, p_{m-1} \Rightarrow \emptyset) = \begin{cases} p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{m-1} \rightarrow 0) \dots)), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \end{cases}$$

$$\rho(p_0) = \{\emptyset \Rightarrow p_0\}.$$

That is, the following conditions are satisfied:

- (i) For every  $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}, \Sigma \vdash_{\widehat{IPC}^* \setminus c} \varphi$  iff  $\{\rho(\sigma) : \sigma \in \Sigma\} \vdash_{\mathbf{FL}_{ew}} \rho(\varphi)$ .
- (ii) For every  $\Gamma \Rightarrow \Delta \in Seq_{\mathcal{L}}^{(\omega, \{0,1\})}, \Gamma \Rightarrow \Delta \dashv\vdash_{\mathbf{FL}_{ew}} \rho\tau(\Gamma \Rightarrow \Delta)$ .

□

**Theorem 3.4.**  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$  is algebraizable with equivalent variety semantics the variety  $\mathbb{RL}$  with translations  $\tau$  from  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$  to  $\mathcal{S}_{\mathbb{RL}}$  (the equational system associated to  $\mathbb{RL}$ ) and  $\rho$  from  $\mathcal{S}_{\mathbb{RL}}$  to  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$  defined as follows:

$$\tau_{(m,1)}(p_0, \dots, p_{m-1} \Rightarrow q_0) = \begin{cases} p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{m-1} \rightarrow q_0) \dots)) \approx 1, & \text{if } m \geq 1 \\ q_0 \approx 1, & \text{if } m = 0 \end{cases}$$

$$\tau_{(m,0)}(p_0, \dots, p_{m-1} \Rightarrow \emptyset) = \begin{cases} p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{m-1} \rightarrow 0) \dots)) \approx 1, & \text{if } m \geq 1 \\ 0 \approx 1, & \text{if } m = 0 \end{cases}$$

$$\rho(p_0 \approx p_1) = \{p_0 \Rightarrow p_1; p_1 \Rightarrow p_0\}$$

**Note 3.5.** Observe that since in  $\mathbb{RL}$  the following holds

- $(x_0 * x_1 * \dots * x_n) \rightarrow y \approx x_0 \rightarrow (x_1 \rightarrow (\dots \rightarrow (x_n \rightarrow y) \dots))$
- $x \vee y \approx y$  iff  $x \leq y$  iff  $x \rightarrow y \approx 1$ ,

then in Theorem 3.4 we could take the following translations:

$$\tau_{(m,1)}(p_0, \dots, p_{m-1} \Rightarrow q_0) = \{(p_0 * p_1 * \dots * p_{m-1}) \vee q_0 \approx q_0\}$$

$$\tau_{(m,0)}(p_0, \dots, p_{m-1} \Rightarrow \emptyset) = \{p_0 * p_1 * \dots * p_{m-1} \approx 0\}.$$

## 4 The varieties $\mathbb{CIBPM}^{sl}$ and $\mathbb{CIBPM}^l$

In this section we study the class of commutative integral bounded semilatticed pseudocomplemented monoids ( $\mathbb{CIBPM}^{sl}$  for short) and the class of commutative integral bounded semilatticed pseudocomplemented monoids ( $\mathbb{CIBPM}^l$  for short). We introduce the notion of pseudocomplement applied to the operation  $*$  of the commutative integral bounded semilatticed monoids. This notion is a generalization of the concept of pseudocomplement defined traditionally in the context of the bounded distributive lattices (see for instance [3]).

**Definition 4.1.** A COMMUTATIVE SEMILATTICED MONOID (commutative  $sl$ -monoid for short) is an algebra  $\mathcal{A} = \langle A, \vee, *, 1 \rangle$  of type  $(2, 2, 0)$  such that:

1.  $\langle A, \vee \rangle$  is a semilattice,
2.  $\langle A, *, 1 \rangle$  is a commutative monoid,
3.  $\mathcal{A} \models (x \vee y) * z \approx (x * z) \vee (y * z)$ .

A COMMUTATIVE LATTICED MONOID (commutative  $l$ -monoid for short) is an algebra  $\mathcal{A} = \langle A, \vee, \wedge, *, 1 \rangle$  of type  $(2, 2, 2, 0)$  such that  $\langle A, \vee, \wedge \rangle$  is a lattice and such that  $\langle A, \vee, *, 1 \rangle$  is a commutative  $sl$ -monoid.

**Definition 4.2.** An algebra  $\mathcal{A} = \langle A, \vee, *, 0, 1 \rangle$  of type  $(2, 2, 0, 0)$  is a COMMUTATIVE INTEGRAL BOUNDED  $sl$ -MONOID if the following conditions are satisfied:

1.  $\langle A, \vee, *, 1 \rangle$  is a commutative  $sl$ -monoid,
2.  $\mathcal{A} \models 0 \vee x \approx x$  (i.e.  $0 \leq x$ ),
3.  $\mathcal{A} \models x \vee 1 \approx 1$  (i.e.  $x \leq 1$ ).

An algebra  $\mathcal{A} = \langle A, \vee, \wedge, *, 0, 1 \rangle$  of type  $(2, 2, 2, 0, 0)$  is a COMMUTATIVE INTEGRAL BOUNDED  $\ell$ -MONOID if  $\langle A, \vee, \wedge \rangle$  is a lattice and  $\langle A, \vee, *, 0, 1 \rangle$  is a commutative integral bounded  $sl$ -monoid.

We will denote by  $\mathbf{CIBM}^{sl}$  the class of commutative integral bounded  $sl$ -monoids and by  $\mathbf{CIBM}^\ell$  the class of commutative integral bounded  $\ell$ -monoids. Obviously these classes of algebras are varieties. The variety of bounded distributive lattices,  $\mathbf{BDL}$  for short, is the subvariety of  $\mathbf{CIBM}^{sl}$  defined by the equation  $x * x \approx x$ .

**Theorem 4.3.** *The variety  $\mathbf{BDL}$  of bounded distributive lattices is the subvariety of  $\mathbf{CIBM}^{sl}$  defined by the equation  $x * x \approx x$ .*

Next we will introduce the notion of *pseudocomplement* with respect to the monoidal operation of  $\mathcal{A} \in \mathbf{CIBM}^{sl}$  or  $\mathcal{A} \in \mathbf{CIBM}^\ell$ . This notion is a generalization of the notion of pseudocomplement in the context of meet-semilattices. Let us recall this definition for distributive lattices with minimum:

**Definition 4.4.** [3, Definition 1, p. 152] *Let  $\mathcal{A} = \langle A, \wedge, \vee, 0 \rangle$  be a distributive lattice with minimum element 0. An element  $a \in A$  is called PSEUDOCOMPLEMENTED if the set  $\{b \in A : a \wedge b = 0\}$  has a maximum. In this case this maximum is denoted by  $\neg a$  and is called the PSEUDOCOMPLEMENT of  $a$ . A PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICE is a distributive lattice with minimum element 0 in which every element has a pseudocomplement.*

Let us recall that if  $\mathcal{A}$  is a pseudocomplemented distributive lattice, then  $a \leq \neg 0$  for all  $a \in A$ , so  $A$  has a maximum element  $1 = \neg 0$ . Hence we can define the class of pseudocomplemented distributive lattices, which we denote by  $\mathbf{PDL}$ , in the language  $\{\vee, \wedge, \neg, 0, 1\}$ .

**Definition 4.5.** *Let  $\lambda \in \{sl, \ell\}$  and  $\mathcal{A} \in \mathbf{CIBM}^\lambda$ . An element  $a \in A$  is called  $*$ -PSEUDOCOMPLEMENTED if the set  $\{b \in A : a * b = 0\}$  has a maximum with respect to the order of the semilattice. In this case this maximum is denoted by  $\neg a$  and is called the  $*$ -PSEUDOCOMPLEMENT of  $a$ .*

**Definition 4.6.** *The class of COMMUTATIVE INTEGRAL BOUNDED PSEUDOCOMPLEMENTED  $\lambda$ -MONOIDS,  $\mathbf{CIBPM}^\lambda$  for short, is the class of algebras  $\mathcal{B} = \langle A, \neg \rangle$  such that:*

1.  $\mathcal{A} \in \mathbf{CIBM}^\lambda$
2. For every  $a \in A$ ,  $\neg a = \max \{b : a * b = 0\}$

Obviously the classes  $\mathbf{CIBPM}^{sl}$  and  $\mathbf{CIBPM}^\ell$  are quasivarieties and in fact they are also varieties. We thank Roberto Cignoli for his personal communication stating this result.

**Theorem 4.7.** *The class  $\mathbf{CIBPM}^{sl}$  is the equational class of algebras  $\mathcal{A} = \langle A, \vee, *, \neg, 0, 1 \rangle$  of type  $(2, 2, 1, 0, 0)$  satisfying the following equations:*

- (1) *The set of equations defining the commutative integral bounded  $sl$ -monoids.*
- (2)  $\neg 1 \approx 0$ .
- (3)  $\neg 0 \approx 1$ .
- (4)  $(x * \neg(y * x)) \vee \neg y \approx \neg y$ .

As an immediate consequence of Theorem 4.7 we obtain an equational definition of the class  $\mathbf{CIBPM}^\ell$ .

**Theorem 4.8.** *The class  $\mathbf{CIBPM}^\ell$  is the equational class of algebras  $\mathcal{A} = \langle A, \vee, \wedge, *, \neg, 0, 1 \rangle$  of type  $(2, 2, 2, 1, 0, 0)$  satisfying the set of equations defining the commutative integral bounded  $\ell$ -monoids and the equations (2), (3) and (4) in Theorem 4.7.*

In the next result we characterize the variety of pseudocomplemented distributive lattices as a subvariety of  $\mathbf{CIBPM}^{s\ell}$ .

**Theorem 4.9.** *The variety  $\mathbf{PDL}$  of pseudocomplemented distributive lattices can be defined as the subvariety of  $\mathbf{CIBPM}^{s\ell}$  defined by the equation  $x * x \approx x$ .*

It is well known that every pseudocomplemented distributive lattice is isomorphic to a subreduct of a Heyting algebra [4, Proof of Theorem 2.6]. The following two theorems are generalizations of this result.

**Theorem 4.10.** *Every  $\mathbf{CIBPM}^{s\ell}$ -algebra is embeddable into a complete residuated lattice. Thus, the class  $\mathbf{CIBPM}^{s\ell}$  is the closure under isomorphism and subalgebras of the class of algebras  $\{\mathcal{A} : \mathcal{A} = \langle A, \vee, *, \neg, 0, 1 \rangle$  and there are operations  $\wedge, \rightarrow$  such that  $\langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle \in \mathbf{RL}\}$ .*

**Theorem 4.11.** *Every  $\mathbf{CIBPM}^\ell$ -algebra is embeddable into a complete residuated lattice. Therefore, the class  $\mathbf{CIBPM}^\ell$  is the closure under isomorphism and subalgebras of the class of algebras  $\{\mathcal{A} : \mathcal{A} = \langle A, \vee, \wedge, *, \neg, 0, 1 \rangle$  and there is an operation  $\rightarrow$  such that  $\langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle \in \mathbf{RL}\}$ .*

The following theorem concerns decidability.

**Theorem 4.12.** *The quasiequational theories of  $\mathbf{CIBPM}^{s\ell}$  and  $\mathbf{CIBPM}^\ell$  are decidable.*

We also show that  $\mathbf{CIBPM}^{s\ell}$  and  $\mathbf{CIBPM}^\ell$  are not the equivalent algebraic semantics of any deductive system.

**Theorem 4.13.** *Let  $\lambda \in \{s\ell, \ell\}$ . Let  $\models_{\mathbf{CIBPM}^\lambda}$  be the equational consequence relation determined by  $\mathbf{CIBPM}^\lambda$ . Then  $\mathcal{S}_{\mathbf{CIBPM}^\lambda} = \langle \mathcal{L}, \models_{\mathbf{CIBPM}^\lambda} \rangle$  is not the equivalent algebraic semantics of any deductive system.*

## 5 Algebraization of the Gentzen systems determined by the sequent calculi $\mathbf{FL}_{\mathbf{ew}}[\vee, *]$ and $\mathbf{FL}_{\mathbf{ew}}[\vee, \wedge, *]$

In the following result we show the equivalence between  $\mathcal{G}_{\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]}$  and  $\mathcal{S}_{\mathbf{CIBPM}^{s\ell}}$  as Gentzen systems (see Section 1.5) by means of the translations  $\tau$  and  $\rho$  defined in Theorem 3.4 but writing  $\tau$  in the form considered in Note 3.5. That is, we show the algebraization of  $\mathcal{G}_{\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]}$ .

**Theorem 5.1.** *The Gentzen system  $\mathcal{G}_{\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]}$  determined by  $\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]$  is algebraizable, with equivalent algebraic semantics the variety  $\mathbf{CIBPM}^{s\ell}$ , with translations  $\tau$  from  $\mathcal{G}_{\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]}$  to  $\mathcal{S}_{\mathbf{CIBPM}^{s\ell}}$  and  $\rho$  from  $\mathcal{S}_{\mathbf{CIBPM}^{s\ell}}$  to  $\mathcal{G}_{\mathbf{FL}_{\mathbf{ew}}[\vee, *, \neg]}$  defined as follows:*

$$\tau_{(m,1)}(p_0, \dots, p_{m-1} \Rightarrow q_0) = \begin{cases} (p_0 * \dots * p_{m-1}) \vee q_0 \approx q_0, & \text{if } m \geq 1 \\ 1 \approx q_0, & \text{if } m = 0 \end{cases}$$

$$\tau_{(m,0)}(p_0, \dots, p_{m-1} \Rightarrow \emptyset) = \begin{cases} p_0 * \dots * p_{m-1} \approx 0, & \text{if } m \geq 1 \\ 1 \approx 0, & \text{if } m = 0 \end{cases}$$

$$\rho(p_0 \approx p_1) = \{p_0 \Rightarrow p_1, p_1 \Rightarrow p_0\}$$

We also show that the result obtained in [15] (see also [14]) stating that the variety of pseudocomplemented distributive lattices is the equivalent algebraic semantics of the Gentzen system determined by the calculus obtained by deleting the implication rules from  $LJ$  (this Gentzen system is essentially the same as  $\mathcal{G}_{\mathbf{FL}_{\text{ewc}}[\vee, *, \neg]}$ ) can be easily obtained from Theorem 5.1.

**Theorem 5.2.** *The Gentzen system  $\mathcal{G}_{\mathbf{FL}_{\text{ewc}}[\vee, *, \neg]}$  determined by the calculus  $\mathbf{FL}_{\text{ewc}}[\vee, *, \neg]$  is algebraizable, with equivalent algebraic semantics the variety  $\mathbf{PDL}$ , with the translations  $\tau$  from  $\mathcal{G}_{\mathbf{FL}_{\text{ewc}}[\vee, *, \neg]}$  to  $\mathcal{S}_{\mathbf{PDL}}$  and  $\rho$  from  $\mathcal{S}_{\mathbf{PDL}}$  to  $\mathcal{G}_{\mathbf{FL}_{\text{ewc}}[\vee, *, \neg]}$  defined in Theorem 5.1.*

We also have that the variety  $\mathbf{CIBPM}^\ell$  is the equivalent algebraic semantics of the Gentzen system  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$ .

**Theorem 5.3.** *The Gentzen system  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  is algebraizable and the variety  $\mathbf{CIBPM}^\ell$  is its equivalent algebraic semantics, with the translations  $\tau$  from  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  to  $\mathcal{S}_{\mathbf{CIBPM}^\ell}$  and  $\rho$  from  $\mathcal{S}_{\mathbf{CIBPM}^\ell}$  to  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  defined in Theorem 5.1.*

To finalize this section we will state two straightforward consequences of the algebraization of  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$ . The first one concerns the contraction rule.

**Theorem 5.4.** *The contraction rule is not admissible in  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$ .*

The second consequence concerns the impossibility of obtaining equivalent Hilbert-style formulations of the Gentzen systems  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$ . Using Theorem 4.13 and the algebraization results we obtain:

**Theorem 5.5.** *The Gentzen systems  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  are not equivalent to any deductive system.*

## 6 The external deductive systems associated to $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$ and $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$

In this section we study the external deductive systems associated to the Gentzen systems  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  denoted here by  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$ , respectively, and we study their position in the Abstract Algebraic Logic hierarchy. First let us recall the definition of external deductive system associated to a Gentzen system.

**Definition 6.1.** *The EXTERNAL DEDUCTIVE SYSTEM<sup>2</sup> associated to a Gentzen system  $\mathcal{G}$  of type  $(\alpha, \beta)$ , with  $0 \in \alpha$  and  $1 \in \beta$ , is the deductive system  $\mathcal{S}_e(\mathcal{G}) = \langle \mathcal{F}m_{\mathcal{L}}, \vdash_{\mathcal{S}_e(\mathcal{G})} \rangle$  defined in the following way: for all  $\Sigma \cup \{\varphi\} \subseteq \mathcal{F}m_{\mathcal{L}}$ ,  $\Sigma \vdash_{\mathcal{S}_e(\mathcal{G})} \varphi$  iff there is a finite subset  $\{\varphi_1, \dots, \varphi_n\} \subseteq \Sigma$  such that  $\emptyset \Rightarrow \varphi_1, \dots, \emptyset \Rightarrow \varphi_n \vdash_{\mathcal{G}} \emptyset \Rightarrow \varphi$ .*

To show that  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are fragments of  $\widehat{IPC}^* \setminus c$  we show first that  $\widehat{IPC}^* \setminus c$  is the external deductive system of  $\mathbf{FL}_{\text{ew}}$  and using this result, the algebraization theorems and the results about subreducts, we show that the Gentzen systems  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  and  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  are fragments of  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ .

<sup>2</sup>We use the name *extern* for this deductive system following A.Avron (see for instance [2]).

**Lemma 6.2.**  $\widehat{IPC}^* \setminus c$  is the external deductive system of  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ .

**Lemma 6.3.**  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]}$  is the  $\{\vee, *, \neg, 0, 1\}$ -fragment of  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ , that is, if  $\mathcal{L} = \{\vee, *, \neg, 0, 1\}$  and  $T \cup \{\Gamma \Rightarrow \Delta\} \subseteq \text{Seq}_{\mathcal{L}}^{(\omega, \{0,1\})}$ , then:

$$T \vdash_{\mathbf{FL}_{\text{ew}}} \Gamma \Rightarrow \Delta \quad \text{iff} \quad T \vdash_{\mathbf{FL}_{\text{ew}}[\vee, *, \neg]} \Gamma \Rightarrow \Delta.$$

**Lemma 6.4.**  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}[\vee, \wedge, *, \neg]}$  is the  $\{\vee, \wedge, *, \neg, 0, 1\}$ -fragment of  $\mathcal{G}_{\mathbf{FL}_{\text{ew}}}$ .  $\square$

Using the two last results we prove that  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are fragments of  $\widehat{IPC}^* \setminus c$ .

**Theorem 6.5.** For all  $\Sigma \cup \{\varphi\} \subseteq \text{Fm}_{\{\vee, *, \neg, 0, 1\}}$ , we have that

$$\Sigma \vdash_{\widehat{IPC}^* \setminus c} \varphi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee, *, \neg]} \varphi.$$

**Theorem 6.6.** For all  $\Sigma \cup \{\varphi\} \subseteq \text{Fm}_{\{\vee, \wedge, *, \neg, 0, 1\}}$  we have that

$$\Sigma \vdash_{\widehat{IPC}^* \setminus c} \varphi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee, \wedge, *, \neg]} \varphi.$$

Let us denote by  $IPC_{\{\vee, *, \neg, 0, 1\}}$  the implication-less fragment of the intuitionistic propositional logic (we use the symbol  $*$  to the additive conjunction). We prove that  $\mathcal{S}_e[\vee, *, \neg]$ , the fragment without conjunction and without implication of  $\widehat{IPC}^* \setminus c$ , is a proper subsystem of  $IPC_{\{\vee, *, \neg, 0, 1\}}$ .

**Theorem 6.7.**  $\mathcal{S}_e[\vee, *, \neg] \subsetneq IPC_{\{\vee, *, \neg, 0, 1\}}$ .

Let us denote by  $IPC_{\{\vee, \wedge, *, \neg, 0, 1\}}^*$  the implication-less fragment of the intuitionistic propositional logic (with  $*$  =  $\wedge$ ). We then obtain the following:

**Theorem 6.8.**  $\mathcal{S}_e[\vee, \wedge, *, \neg] \subsetneq IPC_{\{\vee, \wedge, *, \neg, 0, 1\}}^*$ .  $\square$

Finally we will explain the position of our deductive systems in the Abstract Algebraic Logic hierarchy. As an immediate consequence of Theorem 5.5, we have that the varieties  $\mathbf{CIBPM}^{s\ell}$  and  $\mathbf{CIBPM}^{\ell}$  cannot be the equivalent algebraic semantics of  $\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  respectively. In fact we prove that these external systems are not protoalgebraic. Recall that a deductive system  $\mathcal{S}$  is protoalgebraic if the Leibnitz operator  $\Omega$  is monotonic on the set of  $\mathcal{S}$ -theories (see for example [7]). This condition is equivalent to the fact that there is a set of formulas  $P(p, q)$  (in two variables at most) such that:

$$\begin{aligned} \emptyset \vdash_{\mathcal{S}} P(p, p), & \quad (\text{Reflexivity}) \\ \{p\} \cup P(p, q) \vdash_{\mathcal{S}} q, & \quad (\text{Modus Ponens}) \end{aligned}$$

As a consequence, if  $\mathcal{S}$  is not protoalgebraic, there is no defined binary connective  $\rightarrow$  such that:

$$\begin{aligned} \emptyset \vdash_{\mathcal{S}} p \rightarrow p, & \quad (\text{Identity}) \\ p, p \rightarrow q \vdash_{\mathcal{S}} q, & \quad (\text{Modus Ponens}). \end{aligned}$$

**Theorem 6.9.** Neither the deductive system  $\mathcal{S}_e[\vee, *, \neg]$  nor  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are protoalgebraic.

So these two deductive systems are not algebraizable. Nevertheless it can be seen that the algebraization results for our Gentzen systems give the following completeness statements:

**Theorem 6.10.** *The variety  $\text{CIBPM}^{sl}$  is an algebraic semantics for  $\mathcal{S}_e[\vee, *, \neg]$  with defining equation  $p \approx 1$ .*

**Theorem 6.11.** *The variety  $\text{CIBPM}^l$  is an algebraic semantics for  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  with defining equation  $p \approx 1$ .*

From these results and the decidability of the quasivarieties  $\text{CIBPM}^{sl}$  and  $\text{CIBPM}^l$  (Theorem 4.12) we obtain the following:

**Theorem 6.12.**  *$\mathcal{S}_e[\vee, *, \neg]$  and  $\mathcal{S}_e[\vee, \wedge, *, \neg]$  are decidable.*

## Acknowledgements

The authors thank Roberto Cignoli for his personal communication stating Theorem 4.7 and Francesc Esteve for his suggestions. This study is partially supported by Grant BFM2001-3329 from the Spanish DGES and Grant 2001SGR-00017 from the Generalitat de Catalunya.

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