

On Filter Logics for ‘Most’ and Special Predicates*

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Abstract

Logics for ‘generally’ were introduced for handling, by non-standard generalized quantifiers, assertions with vague notions (important issues in Logic and in Artificial Intelligence). Filter logic is intended to address (some versions) of ‘most’. We show that filter logic can be faithfully embedded into a first-order theory of compatible predicates. We also use representative predicates to eliminate the generalized quantifier. These devices permit using classical first-order methods to reason about consequence in filter logic and help clarifying the role of such logics for ‘generally’.

Keywords Vague notions, generalized quantifiers, filter logic, compatible predicates, representative predicates, interpretation, proof methods.

1 Introduction

In this paper we show that filter logic (FL) can be faithfully embedded into a first-order logic theory of certain predicates. We thus provide a framework where the semantic intuitions of FL (for ‘most’) can be combined with proof methods for classical first-order logic (CFOL). This framework supports theorem proving in FL, as it permits using proof procedures and theorem provers for CFOL. It will also help clarifying the role of such extensions of CFOL for ‘generally’.

Some logics for ‘generally’ (LG) were introduced for handling assertions with some vague notions, such as ‘generally’ and ‘rarely’, by non-standard generalized quantifiers [?, ?].¹ Their expressive power is quite convenient and they have sound and complete deductive systems. This, however, still leaves open the question of theorem proving, namely theorem provers for them. We will show that special predicates (representative and compatible predicates) allow one to use existing theorem provers (for CFOL) for this task.²

This paper is structured as follows. In section 2 we review some ideas about LG. Section 3 introduces representative sets and functions. In section 4 we examine the translation of ∇ by predicates and compatibility. Section 5 introduces representative axioms to internalize our translation. Section 6 exhibits a framework for reducing filter reasoning to CFOL theories of compatible predicates and examines some applications. Section 6 contains some concluding remarks about our approach.

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¹These logics are related to variants of default logic [?], but they are quite different logical systems, both technically and in terms of intended interpretations [?]. The expressive power of our generalized quantifiers [?] paves the way for other possible applications where it may be helpful: e. g. expressing some fuzzy concepts [?].

²We will concentrate on FL (for ‘most’), but the main lines can be adapted to some other LG (cf. [?, ?]).

2 Logics for Reasoning about ‘Generally’

We now examine some ideas about LG: motivations and some technical aspect

We first briefly consider some motivations underlying LG [?, ?]. Assertions and arguments involving some vague notions, such as ‘generally’, ‘rarely’, ‘most’, ‘several’, etc., occur often in ordinary language and in some branches of science. One often meets assertions such as “Several bodies expand when heated”, “Most birds fly” and “Metals rarely are liquid under ordinary conditions”. The assertions “Whoever likes sports watches SporTV” and “Boys generally like sports” appear to lead to “Boys generally watch SporTV”. Qualitative reasoning about such notions often occurs in everyday life. We can express some assertions in CFOL: e. g. “All birds fly” and “Some birds fly”. But, what about vague assertions like “Birds generally fly”? We wish to express such assertions and reason about them in a precise manner. Logics for some vague notions can be obtained by extending CFOL with an operator ∇ and axioms [?]. So, one can express “Birds generally fly” by $\nabla vF(v)$.

Logics for ‘generally’ extend CFOL by generalized quantifiers, interpreted as ‘generally’ [?, ?]. We now briefly review FL. We use $L(\rho)$ for the usual CFOL language (with equality \simeq) of a given signature ρ . It is convenient to have a fixed, though arbitrary, ordering for the set V of variables: in each list of variables, they will be listed according to this ordering. We will use $L^\nabla(\rho)$ for the extension of $L(\rho)$ by the new operator ∇ . The formulas of $L^\nabla(\rho)$ are built by the usual formation rules and a new variable-binding formation rule giving *generalized formulas*: for each variable v , if φ is a formula in $L^\nabla(\rho)$ then so is φ . Other syntactic notions, like substitution ($\varphi[v/t]$), can be easily adapted.

We provide semantic interpretation for ‘generally’ by enriching structures with filters and extending the definition of satisfaction to ∇ . A *filter structure* $\mathcal{A}^\mathcal{F} = \langle \mathcal{A}, \mathcal{F} \rangle$ for signature ρ consists of a usual structure \mathcal{A} for ρ together with a filter \mathcal{F} over the universe A of \mathcal{A} . We extend the usual definition of *satisfaction* of a formula in a structure under assignment \underline{a} to its (free) variables, using the extension $\mathcal{A}^\mathcal{F}[\varphi(\underline{a}, z)] := \{b \in A : \mathcal{A}^\mathcal{F} \models \varphi(\underline{a}, z)[\underline{a}, b]\}$, as follows: $\mathcal{A}^\mathcal{F} \models \nabla z\varphi(\underline{a}, v)[\underline{a}]$ iff $\mathcal{A}^\mathcal{U}[\varphi(\underline{a}, z)]$ is in \mathcal{F} . Other semantic notions, such as reduct, model ($\mathcal{A}^\mathcal{F} \models \Gamma$) and validity, are as usual [?, ?].³ The notion of *filter consequence* is as expected: $\Gamma \models^\mathcal{F} \tau$ iff $\mathcal{A}^\mathcal{F} \models \tau$, for every filter model $\mathcal{A}^\mathcal{F} \models \Gamma$.

We can formulate deductive systems for our LG by adding schemas to a calculus for CFOL. To set up a deductive system \vdash^f for FL, we take a sound and complete deductive calculus for CFOL, with Modus Ponens as the sole inference rule (as in [?]), and extend its set Λ_ρ of axiom schemas by adding a set Δ^ρ of new axiom schemas (coding properties of filters), to form $\Lambda^\rho := \Lambda_\rho \cup \Delta^\rho$. This set Δ^ρ consists of all the universal generalizations of the following five schemas (where φ , ψ and θ are formulas of $L^\nabla(\rho)$):

$$\begin{array}{ll} [\nabla\exists] \quad \nabla v\varphi \rightarrow \exists v\varphi & [\forall\nabla] \quad \forall v\varphi \rightarrow \nabla v\varphi \\ [\nabla\alpha] \quad \nabla v\varphi \leftrightarrow \nabla u\varphi[v := u] & \text{for a new variable } u \text{ not occurring in } \varphi \\ [\rightarrow\nabla] \quad \forall v(\psi \rightarrow \theta) \rightarrow (\nabla v\psi \rightarrow \nabla v\theta) & [\nabla\wedge] \quad (\nabla v\psi \wedge \nabla v\theta) \rightarrow \nabla v(\psi \wedge \theta) \end{array}$$

These schemas express properties of filters, with $[\nabla\alpha]$ covering alphabetic variants. Derivations are CFOL derivations from the schemas: $\Gamma \vdash^f \varphi$ iff $\Gamma \cup \Lambda^\rho \vdash \varphi$. Other usual deductive notions, such as (maximal) consistent sets, witnesses and conservative extension [?, ?], can be easily adapted.

We have a sound and complete deductive system for our logic ($\models^\mathcal{F} = \vdash^f$), which is a proper conservative extension of CFOL [?, ?, ?].⁴ In the sequel, we show that we can reason about such ‘most’-assertions

³Satisfaction of a formula hinges only on the realizations assigned to its symbols.

⁴Soundness is clear and completeness can be established by adapting Henkin’s familiar proof for CFOL. It is not

entirely within CFOL by means of special predicates: we will show that FL can be faithfully embedded into a CFOL theory of compatible predicates.⁵

3 Representative Sets and Functions

We will now introduce the notions of representative sets and functions.

We first examine some ideas about representative sets. “Most birds fly” ($\nabla vF(v)$) is intended to mean that the set of flying birds is an ‘important’ set of birds. As such, we cannot infer from it that an arbitrary bird flies (we do not have instantiations: $\nabla z\varphi \rightarrow \varphi[z/t]$ is not valid). We may consider as typical a bird with the properties that most birds have; so, we can instantiate generalized formulas to typical objects: for each typical bird b (if any) we have $\nabla vF(v) \rightarrow F(b)$. Such typical objects are somewhat elusive: they may fail to exist. (What would be a typical natural number? Considering ‘most’ as cofinite, no standard natural can be typical.) Now, what about the converse: from which set of birds can we infer $\nabla vF(v)$? This appears problematic, like experimental induction. Let us call a set S of birds sufficiently representative when one can infer $\nabla vF(v)$ from $F(b)$, for all $b \in S$. These ideas motivate calling a set S representative (with respect to flying) when $\nabla vF(v)$ is equivalent to $F(b)$, for all $b \in S$.

We now examine representative objects in a filter structure $\mathcal{A}^{\mathcal{F}}$. A *representative set* for a generalized sentence $\nabla z\varphi$ is a subset $S \subseteq A$ such that $\mathcal{A}^{\mathcal{K}} \models \nabla z\varphi$ iff, for every $a \in S$, $\mathcal{A}^{\mathcal{K}} \models \varphi[a]$. We can extend this idea to representative functions for formulas with free variables. A *representative function* for a generalized formula $\nabla z\varphi$ with m free variable is an m -ary function $r : A^m \rightarrow \wp(A)$, assigning to each m -tuple $\underline{a} \in A^m$ a representative set $r(\underline{a}) \subseteq A$: $\mathcal{A}^{\mathcal{K}} \models \nabla z\varphi[\underline{a}]$ iff, for every $b \in r(\underline{a})$, $\mathcal{A}^{\mathcal{K}} \models \varphi[\underline{a}, b]$.⁶

In the sequel, we will show that we can translate ‘most’-assertions by means of special predicates. We will use expansions of signatures by new predicates. Given a signature σ , consider for each $n \in \mathbb{N}$, a new $(n + 1)$ -ary predicate symbol p_n not in σ , and form the expansion $\sigma[P] := \sigma \cup P$, obtained by adding the set $P := \{p_n : n \in \mathbb{N}\}$. We will translate ∇ by universal relativizations to these predicates.

4 Translation of ‘Most’ to First-Order

We will now consider the translation of ∇ by predicates and examine the idea of compatibility.

We translate ∇ by universal relativizations to new predicate symbols in the set $P := \{p_n : n \in \mathbb{N}\}$. We transform $\nabla z\theta(\underline{u}, z)$, with list \underline{u} of m free variables, to $(\forall z : p_m[\underline{u}])\theta(\underline{u}, z)$, which abbreviates $\forall z[p_m(\underline{u}, z) \rightarrow \theta(\underline{u}, z)]$. Now, we eliminate ∇ from a formula φ of $L^{\nabla}(\sigma[P])$ by replacing (inside-out) each subformula $\nabla z\theta(\underline{u}, z)$ of φ by $(\forall z : p_{|\underline{u}|}(\underline{u}))\theta(\underline{u}, z)$ to obtain a formula of $L(\sigma[P])$: its P -transform φ_P . This process defines a ∇ -eliminating P -translation $(\)_P : L^{\nabla}(\sigma[P]) \rightarrow L(\sigma[P])$.

We now introduce compatibility. Our transformation should preserve provability. For this purpose, we need some (FO) information connecting the new predicate symbols: their compatibility. A motivation for compatibility comes from examining the translation of the filter schemas into FO: each instance of the schemas $[\forall\nabla]$ and $[\nabla\alpha]$ becomes logically valid, but no so for the other schemas. Compatibility

difficult to see that we have conservative extensions of CFOL. These extensions are proper, because some sentences, such as $\exists u\nabla zu \simeq z$, cannot be expressed without ∇ [?, ?].

⁵The crucial issue here is having a faithful interpretation. An interpretation into CFOL can be obtained by replacing throughout ∇ by \forall : each filter axiom is translated to a logical validity, but this interpretation is not faithful.

⁶We may view an m -ary function $r : A^m \rightarrow \wp(A)$ as an $(m + 1)$ -ary relation $r \subseteq A^m \times A$ given by $\langle \underline{a}, b \rangle \in r$ iff $b \in r(\underline{a})$.

will provide a way to handle schemas $[\nabla\exists]$, $[\nabla\wedge]$ and $[\rightarrow\nabla]$. Our requirements are related to the P -translation of generalized formulas: $(\nabla z\varphi)P := (\forall z : p_m|\underline{u})\varphi_P$, for $\nabla z\varphi$ with list \underline{u} of m free variables. We also consider the list \underline{v}_n of the n variables v_1, \dots, v_n , for each $n \in N$.

The inclusion in the translation is vacuously satisfied by a void predicate. So, our first requirement $\underline{p}_n(a) \neq \emptyset$ is the *non-voidness axiom* $\exists(p_n) : \forall \underline{v}_n \exists v_0 p_n(\underline{v}_n, v_0)$. The intuition of having a representative set for each n -tuple suggests a second compatibility requirement, $\underline{p}_{n+1}(a, b) \subseteq \underline{p}_n(a)$, expressed by the *decreasing axiom* $\downarrow(p_n) : \forall \underline{v}_n \forall v_{n+1} \forall v_0 [p_{n+1}(\underline{v}_n, v_{n+1}, v_0) \rightarrow p_n(\underline{v}_n, v_0)]$. The third requirement mirrors the idea that new free variables do not matter, as expressed by the *restriction axiom* $(p_n^{\underline{v}}\varphi) : \forall \underline{u} \forall \underline{v} [\forall z (p_{m+k}(\underline{u}, \underline{v}, z) \rightarrow \varphi) \rightarrow \forall z (p_m(\underline{u}, z) \rightarrow \varphi)]$ (where \underline{v} is a list of k variables other than \underline{u} and z).

We will extend these axioms to schemas for sets. The *non-voidness* and *decreasing schemas* are the sets $\exists[P] := \{\exists(p_n) : p_n \in P\}$ and $\downarrow[P] := \{\downarrow(p_n) : p_n \in P\}$, respectively. The *restriction schema* for $L^\nabla(\sigma)$ is the set $[P \uparrow \sigma^\nabla]$ consisting of the restriction axioms for the P -translation of every generalized formula $\nabla z\varphi$ in $L^\nabla(\sigma)$. We shall also use the *compatibility schema* $\Omega^* := \exists[P] \cup \downarrow[P] \cup [P \uparrow \sigma^\nabla]$.

We now examine some properties of compatible predicates.

Lemma 1 *The P -translations of the filter axioms in Δ^σ follow from the compatibility schema Ω^* .*

Proof The schema $\exists[P]$ yields the translation of $[\nabla\exists]$ and the P -translation of each axiom in $[\nabla\wedge]$ and $[\rightarrow\nabla]$ follows from the other two schemas (which permit adjusting the free variable).

We see that our translation interprets FL into a CFOL theory of compatible predicates.

Proposition 2 *The P -translation $(\)_P$ restricted to $L^\nabla(\sigma)$ interprets FL (for signature σ) into any CFOL theory $\Sigma \subseteq L(\sigma[P])$ where the new predicate symbols in P are compatible (i. e. $\Omega^* \subseteq Cn(\Sigma)$): for each set $\Gamma \cup \{\tau\}$ of sentences of $L^\nabla(\sigma)$, if $\Gamma \vdash^f \tau$ then $\Sigma \cup \Gamma_P \vdash \tau_P$.*

Proof The assertion follows from the preceding lemma, giving $[\Delta^\sigma]_P \subseteq Cn(\Omega^*)$.

5 Representative Predicates and Axioms

We now introduce representative axioms to internalize our translation of ∇ by special predicates.

We first formulate representative predicates by sentences in the expanded language $L^\nabla(\sigma[P])$.

Given a generalized formula $\nabla z\varphi$ of $L^\nabla(\sigma[P])$ with list \underline{u} of m free variables, the formula $(\forall z : p_m|\underline{u})\varphi$ is in $L^\nabla(\sigma[P])$, and the *representative axiom* $\partial(p_m\nabla z\varphi)$ for $\nabla z\varphi$ is the universal closure of the formula $\nabla z\varphi \leftrightarrow (\forall z : p_m|\underline{u})\varphi$ of $L^\nabla(\sigma[P])$. We extend this idea to sets of formulas. The *representative schema* for a set Ψ of formulas of $L^\nabla(\sigma[P])$ is the set $\partial[\Psi]$ consisting of the representative axioms for every generalized formula $\nabla z\varphi$ in Ψ . When Ψ is the set of all the (generalized) formulas of signature σ , we use $\partial[\sigma^\nabla] := \partial[L^\nabla(\sigma)]$ for the *representative schema* for $L^\nabla(\sigma)$.

We can now see that the representative axioms internalize our P -translation $(\)_P$ into $L(\sigma[P])$.

Proposition 3 *The representative schema $\partial[\Psi]$ for a set Ψ of formulas of $L^\nabla(\sigma[P])$, closed under subformulas, yields the equivalence between a formula ψ in Ψ and its P -transform ψ_P : $\partial[\Psi] \vdash^f \psi \leftrightarrow \psi_P$, for $\psi \in \Psi$.*

Proof By induction on the structure of the formulas.

Thus, representative axioms enable the elimination of the new quantifier ∇ in favor of representative predicates. In particular, $\partial[\sigma^\nabla] \vdash^f \tau \leftrightarrow \tau_P$, for every sentence τ of $\mathbb{L}^\nabla(\sigma)$. We can now see that the restriction axioms are filter consequences of the representative schema.

Corollary 4 *The restriction axioms in $[P \uparrow \sigma^\nabla]$ are filter consequences of the representative schema $\partial[\sigma^\nabla]$.*

Proof Each restriction axiom $(p_n^v \varphi)$ is a filter consequence of two representative axioms.

We will now establish the conservativeness of extensions by compatible and representative axioms.

Representative sets are elusive: many structures fail to have them. Nevertheless, we can add them conservatively into theories. To establish this fact, we introduce some terminology and show that a filter structure has compatible relations that are representative for a finite set of generalized formulas.

Consider a filter structure $\mathcal{A}^\mathcal{F} = \langle \mathcal{A}, \mathcal{F} \rangle$. We call an n -ary function $\underline{r} : A^n \rightarrow \wp(A)$ *strong* iff $\underline{r}(\underline{a}) \in \mathcal{F}$, for every $\underline{a} \in A^n$. Given an $(n+1)$ -ary function $\underline{s} : A^{n+1} \rightarrow \wp(A)$, we call \underline{s} *inclusive* with respect to n -ary function \underline{r} iff, for every $\langle a_1, \dots, a_n, b \rangle \in A^{n+1}$, $\underline{s}(a_1, \dots, a_n, b) \subseteq \underline{r}(a_1, \dots, a_n)$. We will call a set R of relations on A *adequate* iff the associated functions into $\wp(A)$ are strong and inclusive.

A filter structure has adequate relations that are representative for a finite set of formulas.

Lemma 5 *A filter structure $\mathcal{A}^\mathcal{F}$ has adequate relations that are representative for a finite set Φ of formulas.*

Proof The functions are intersections of extensions in \mathcal{F} : with v_m as the highest free variable in the generalized formulas of Φ , take $\underline{r}_n(\underline{a})$ as the intersection of the extensions $\mathcal{A}^\mathcal{F}[\varphi(\underline{a}, z)]$ in \mathcal{F} for $\nabla z \varphi$ in Φ with free variables up to v_n , for $0 \leq n \leq m$. The associated relations are adequate and representative.

So, one can always conservatively extend a theory by compatible and representative predicates.

Proposition 6 *Given a set Γ of sentences of $\mathbb{L}^\nabla(\sigma)$, the theory $\Gamma^*[\partial] := \Gamma \cup \Omega^* \cup \partial[\sigma^\nabla] \subseteq \mathbb{L}^\nabla(\sigma[P])$ is a conservative extension of $\Gamma \subseteq \mathbb{L}^\nabla(\sigma)$: $\Gamma^*[\partial] \vdash^f \tau$ iff $\Gamma \vdash^f \tau$, for each sentence $\tau \in \mathbb{L}^\nabla(\sigma)$.*

Proof The assertion follows from the previous lemma and corollary: $\Gamma \cup \exists[P] \cup \downarrow [P] \cup \partial[\sigma^\nabla]$ is a conservative extension of Γ (by compactness) and $[P \uparrow \sigma^\nabla] \subseteq Cn^f(\partial[\sigma^\nabla])$.

6 Framework for Reasoning with ‘Most’ in First-Order

Now we exhibit our reduction framework based on a faithful interpretation.

We first put together our results to see that we have a common (conservative) extension.

$$\frac{\frac{\mathbb{L}^\nabla(\sigma)}{\Gamma} \quad \Lambda}{\Gamma \cup \Omega^* \cup \partial[\sigma^\nabla]} \quad \frac{\frac{\mathbb{L}^\nabla(\sigma[P])}{\Gamma_P \cup \Omega^*} \quad \cap}{\Gamma_P \cup \Omega^* \subseteq \mathbb{L}(\sigma[P])} \quad \frac{\Gamma \cup \Omega^* \cup \partial[\sigma^\nabla]}{\mathbb{L}^\nabla(\sigma[P])} \quad \frac{\Gamma_P \cup \Omega^* \subseteq \mathbb{L}(\sigma[P])}{\mathbb{L}^\nabla(\sigma[P])}$$

Corollary 7 *Given a set Γ of sentences of $\mathsf{L}^\nabla(\sigma)$, the theory $\Gamma^*[\partial] := \Gamma \cup \Omega^* \cup \partial[\sigma^\nabla] \subseteq \mathsf{L}^\nabla(\sigma[P])$ is a conservative extension of $\Gamma \subseteq \mathsf{L}^\nabla(\sigma)$ which extends the CFOL theory $\Gamma_P \cup \Omega^* \subseteq \mathsf{L}(\sigma[P])$.*

Proof The assertion follows from the previous results: the propositions in 5.

So, we have a faithful interpretation of FL into the theory of compatible predicates.

Theorem 8 *The P -translation $(\)_P$ restricted to $\mathsf{L}^\nabla(\sigma)$ interprets faithfully FL (for σ) into the CFOL theory $\Omega^* \subseteq \mathsf{L}(\sigma[P])$: $\Gamma \vdash^f \tau$ iff $\Gamma_P \cup \Omega^* \vdash \tau_P$, for a set $\Gamma \cup \{\tau\}$ of sentences of $\mathsf{L}^\nabla(\sigma)$.*

Proof The assertion follows from the previous results: the proposition in 4 and the preceding corollary.

We thus have a sound and complete reduction of filter consequence to CFOL derivability with compatible predicates: establishing $\Gamma \models^{\mathcal{K}} \tau$ amounts to showing that $\Gamma_P \cup \Omega^* \vdash \tau_P$.

We will now examine some examples illustrating the application of our reduction procedure.

As a simple example, we see that $\nabla z \forall u L(u, z) \vdash^f \forall u \nabla z L(u, z)$, since the translated conclusion $[\forall u \nabla z L(u, z)]_P$ is a CFOL consequence of the translated hypothesis $[L(u, z)]_P$ together with the decreasing axiom $\downarrow(p_0)$. Similarly, we can reduce $\exists v \nabla z \varphi \vdash^f \nabla z \exists v \varphi$ to CFOL consequence (from $[P \uparrow \sigma^\nabla]$). We also see that $\{\nabla z \psi, \nabla z(\psi \rightarrow \theta)\} \vdash^f \nabla z \theta$, because $[\nabla z \theta]_P$ is a CFOL consequence of the translated hypotheses $[\nabla z \psi]_P$ and $[\nabla z(\psi \rightarrow \theta)]_P$ together with the schemas $\downarrow[P]$ and $[P \uparrow \sigma^\nabla]$. For an induction-like example, consider a universe of emeralds and imagine every emerald examined to be green. If we also assume that most emeralds are similar to those examined and that similarity generally transfers colors, then we can infer that most emeralds are green: $\nabla z G(z)$. Here Γ consists of $\varepsilon: \forall u(E(u) \rightarrow G(u))$, $\delta: \nabla z \exists u(E(u) \wedge S(u, z))$ and $\gamma: \nabla z \nabla u[(E(u) \wedge S(u, z)) \rightarrow (G(u) \rightarrow G(z))]$.

In many practical cases (as in databases, for instance), we deal only with finitely many formulas. As the preceding examples indicate, in such cases one needs only a finite number of new predicates and axioms related to generalized subformulas of the formulas involved. These ideas are also useful for investigating provability in FL, as we will now illustrate with some cases involving simply generalized formulas: those of the form $\nabla z \varphi$, for φ without ∇ .

Proposition 9 *Consider a set Σ of sentences and a formula ψ in $\mathsf{L}(\sigma)$.*

1. *For each formula $\theta \in \mathsf{L}(\sigma)$: $\Sigma \cup \nabla z \psi \vdash^f \nabla z \theta$ iff $\Sigma \vdash \forall z(\psi \rightarrow \theta)$ and $\Sigma \vdash^f \nabla z \theta$ iff $\Sigma \vdash \forall z \theta$.*
2. *For each sentence $\tau \in \mathsf{L}(\sigma)$: $\Sigma \cup \nabla z \psi \vdash^f \tau$ iff $\Sigma \cup \exists z \psi \vdash \tau$.*
3. *There exists a sentence $\tau \in \mathsf{L}(\sigma)$ so that $\Sigma \vdash^f \nabla z \psi \leftrightarrow \tau$ iff $\Sigma \vdash \exists z \psi \rightarrow \forall z \psi$.*

Proof Reduce to compatible axioms. For (1): (\Rightarrow) , choose compatible functions giving the nonempty z -extension of $\psi \wedge \neg \theta$ in some model $\mathcal{M} \models \Sigma$. Part (2) is like (1) and they yield (3).

7 Conclusion

Logics for ‘generally’ were introduced for handling assertions with vague notions, such as ‘generally’, ‘most’, ‘several’. We have established that FL (filter logic) can be faithfully embedded into a CFOL theory of compatible predicates. This provides a CFOL reduction of filter consequence, thus allowing the use of methods for CFOL. So, there are many proof procedures and theorem provers at one’s disposal [?]. In addition, this approach helps to clarify the place of FL: despite its semantics based on filters, this extension of CFOL can be regarded as (part of) a CFOL theory of compatible predicates.

The development has concentrated on FL (for ‘most’), but its main lines can be adapted to some other LG.⁷ Our framework is not meant as a competitor to non-monotonic logics, although it does solve monotonically various problems (e. g., generic reasoning) addressed by non-monotonic approaches.

As special predicates enable using any available classical proof methods, we expect to have paved the way for theorem proving in FL for ‘most’. In fact, our framework permits combining the intuitions about ‘important’ subsets of the universe, extensions of CFOL by non-standard generalized quantifiers and proof methods for CFOL. Such combinations appear to be very fruitful, deserving further consideration.

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⁷The compatibility axioms in the reduction for FL are somewhat more modular than in the case of ultrafilter logic, where the axioms of the special functions also depend on the source language [?]. Other approaches add a predicate or function symbol for each generalized formula (like Skolem functions).