Abstract Modalities and Institutions

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Abstract.
We define abstract modal semantics using institutions. Modalities can then be generated over a wide variety of logics. Using tools from institution-independent model theory we state a preservation result for the modal satisfaction.

1. Introduction

Institutions (Goguen and Burstall, 1992) formalize the intuitive notion of logical system, using concepts from category theory. Their initial goal was "to do as much computing science as possible" independently of what the underlying logic may be. Recently, this 'institution-independent' approach found applications in model theory (Diaconescu, 2003; Diaconescu, 2004b; Diaconescu, 2004a). Based on the internal logic of (Diaconescu, 2003) where logical connectives and quantifiers can be developed internally to any institution, our paper introduces a Kripke semantics (Kripke, 1959; Goldblatt, 1993) at the level of arbitrary institutions. Also, we develop a fundamental preservation result, namely that each modal sentence is preserved by ultraproducts of frames. For knowledge on institution-independent ultraproducts the reader should consult (Diaconescu, 2003).

This paper gives only the basic definitions and results, the missing proofs, examples, and more discussion can be found in (Diaconescu and Stefaneas, 2003). Our notation and terminology of category and institution theory is consistent to that of (Diaconescu, 2003; Diaconescu, 2004b; Diaconescu, 2004a).

2. Possible worlds semantics in arbitrary institutions

DEFINITION 1. Assume an institution morphism $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ign, Sen, \mathrm{MOD}, \models) \to (\mathbb{S}ign^\Delta, Sen^\Delta, \mathrm{MOD}^\Delta, \models^\Delta)$ from a 'base' institution to a 'domain' institution. Given a signature Σ in $\mathbb{S}ign$, a Σ*-frame* (W,R) consists of

- a family of Σ-models $W\colon I_W \to |\mathrm{MOD}(\Sigma)|$ such that *sharing condition* $\beta^\Delta_\Sigma(W^i) = \beta^\Delta_\Sigma(W^{i'})$ holds for each $i, i' \in I_W$, and
- an binary *accesibility relation* R on the index set I_W.

A Σ*-frame homomorphism* $(h^W, h^I)\colon (W,R) \to (W', R')$ consists of

- a function $h^I : I_W \to I_{W'}$ between the index sets which is a relation homomorphism, i.e. $\langle i, j\rangle \in R$ implies $\langle h^I(i), h^I(j)\rangle \in R'$, and
- a natural transformation $h^W : W \Rightarrow h^I;W'$ such that $\beta^\Delta_\Sigma((h^W)^i) = \beta^\Delta_\Sigma((h^W)^{i'})$ for each $i, i' \in I_W$.

For each K $\in \{T, S4, S5\}$, the Σ-frames and their homomorphisms form a category denoted K^Δ-$\mathrm{MOD}(\Sigma)$. □

In the actual examples various frame concepts correspond to various level of sharing obtained by varying the domain institution. For example, in the framework of first order logic, the most conventional case realising the ides that valuations in each frame interpret uniformly the (first order) variables, is obtained when the domain signatures are sets of sort symbols plus constant declarations.

DEFINITION 2. Given a signature morphism $\varphi\colon \Sigma \to \Sigma'$, each Σ'-frame (W', R') can be reduced to the Σ-frame $(W'; \mathrm{MOD}(\varphi), R')$. Similarly, each frame homomorphism (h^W, h^I) can be reduced to $(h^W \mathrm{MOD}(\varphi), h^I)$. This defines a frame model functor K^Δ-$\mathrm{MOD}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$. $\Box$

The definition below extends the internal logic of (Diaconescu, 2003) with modalities.

DEFINITION 3. Given any institution $(\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$, a functor M-$Sen\colon \mathbb{S}ign \to \mathbb{S}et$ is a *modal sentence functor over Sen* when

– for each signature Σ, each sentence in M-$Sen(\Sigma)$ can be obtained from the sentences of $Sen(\Sigma)$ by iterative application of

- usual logical connectives (i.e. conjunction, disjunction, negation, implication, etc.),
- modal (unary) connectives (i.e. necessity $\Box\_$, possibility $\Diamond\_$),
- universal or existential quantification along signature morphisms, i.e. $(\forall\chi)\rho$ is a Σ-sentence when ρ is a Σ_1-sentence and $\chi\colon \Sigma \to \Sigma_1$ is a signature morphism.

– for each signature morphism $\phi\colon \Sigma \to \Sigma_1$,

- M-$Sen(\phi)(\rho) = Sen(\phi)(\rho)$ when $\rho \in Sen(\Sigma)$,
- M-$Sen(\phi)(\rho_1 \wedge \rho_2) =$ M-$Sen(\phi)(\rho_1) \wedge$ M-$Sen(\phi)(\rho_2)$; similarly for other logical connectives,
- M-$Sen(\phi)(\Box\rho) = \Box($M-$Sen(\phi)(\rho))$; similarly for $\Diamond$, and
- M-$Sen(\phi)((\forall\chi)\rho) = (\forall\chi')$M-$Sen(\phi_1)(\rho)$ where

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi} & \Sigma' \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi'} \\ \Sigma_1 & \xrightarrow[\phi_1]{} & \Sigma'_1 \end{array}$$

is a pushout of signatures; similarly for existential quantification.

$\Box$

DEFINITION 4. Given a signature Σ, for each Σ-frame (W, R) and each modal sentence $\rho \in$ M-$Sen(\Sigma)$ we define the *satisfaction of* ρ *in* (W, R) *at the possible world* $i \in I_W$, denoted $(W, R) \models^i \rho$ by

– $(W, R) \models^i \rho$ if and only if $W^i \models \rho$ when $\rho \in Sen(\Sigma)$,
– $(W, R) \models^i \rho_1 \wedge \rho_2$ if and only if $(W, R) \models^i \rho_1$ and $(W, R) \models^i \rho_2$; similarly for other logical connectives,
– $(W, R) \models^i \Box\rho$ if and only if $(W, R) \models^j \rho$ for each $\langle i, j\rangle \in R$, and
– $(W, R) \models^i (\forall\chi)\rho$ if and only if $(W', R) \models^i \rho$ for each expansion (W', R) of (W, R) along χ.

For each Σ-frame (W, R) and each Σ-modal sentence ρ, then (W, R) *satisfies* ρ, denoted $(W, R) \overset{m}{\models} \rho$, if and only if $(W, R) \models^i \rho$ at each possible world $i \in I_W$. $\Box$

THEOREM 1. Assume that both the base institution and the domain institution are semi-exact and that the mapping between their categories of signatures preserve pushouts.

Then for any modal sentence functor M-Sen over Sen, and for each K $\in \{T, S4, S5\}$, $(\mathbb{S}ign, $M-$Sen, K^\Delta$-$\mathrm{MOD}, \overset{m}{\models})$ is an institution. $\Box$

3. Reduced products of frames

Let $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ign, Sen, \text{MOD}, \models) \to (\mathbb{S}ign^\Delta, Sen^\Delta, \text{MOD}^\Delta, \models^\Delta)$ be an institution morphism from a base institution to a domain institution such that

EB. the base institution $(\mathbb{S}ign, Sen, \text{MOD}, \models)$ is semi-exact,

ED. the domain institution $(\mathbb{S}ign^\Delta, Sen^\Delta, \text{MOD}^\Delta, \models^\Delta)$ is semi-exact,

ET. $\Phi^\Delta \colon \mathbb{S}ign \to \mathbb{S}ign^\Delta$ preserves pushouts, and

RP. for each signature Σ the category of Σ-models $\text{MOD}(\Sigma)$ has small products and directed colimits and β^Δ_Σ preserves them.

LI. for any signature Σ, β^Δ_Σ *lifts isomorphisms*, i.e. if $\beta^\Delta_\Sigma(M)$ is isomorphic to N' there exists N isomorphic to M such that $N' = \beta^\Delta_\Sigma(N)$.

In the actual examples the conditions [EB.], [ED.], and [ET.] as very easy, while the condition [LI.] is almost trivial. Condition [RP.] is also very easy being very similar to the preservation of reduced products by model reducts corresponding to signature morphisms in an institution.

PROPOSITION 1. For each $K \in \{T, S4, S5\}$ and for each signature Σ, the category of frames K^Δ-$\text{MOD}(\Sigma)$ has reduced products. □

Let $(\mathbb{S}ign, \text{M-}Sen, K^\Delta\text{-MOD}, \overset{m}{\models})$ be a modal institution over the institution morphism $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ign, Sen, \text{MOD}, \models) \to (\mathbb{S}ign^\Delta, Sen^\Delta, \text{MOD}^\Delta, \models^\Delta)$ satisfying the conditions [EB.], [ED.], [ET.], [RP.], and [LI.].

DEFINITION 5. Let $\mathcal{F}$ be a class of filters. For a signature Σ, a modal sentence ρ is

- *modally preserved by $\mathcal{F}$-reduced factors* when for each $i \in I_{W_F}$, $\prod_F (W_i, R_i) = (W_F, R_F) \models^i \rho$ implies "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$", and
- *modally preserved by $\mathcal{F}$-reduced products* when for each $i \in I_{W_F}$, "there exists $J \in F$ and $k \in \mu_J^{-1}(i)$ such that $(W_j, R_j) \models^{k_j} \rho$ for each $j \in J$" implies $\prod_F (W_i, R_i) = (W_F, R_F) \models^i \rho$.

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{(W_i, R_i)\}_{i \in I}$ of Σ-frames. □

The following modal fundamental ultraproducts theorem represents a modal extension of the main result of (Diaconescu, 2003).

THEOREM 2. For any modal institution $(\mathbb{S}ign, \text{M-}Sen, K^\Delta\text{-MOD}, \overset{m}{\models})$ over an institution morphism $(\Phi^\Delta, \alpha^\Delta, \beta^\Delta)\colon (\mathbb{S}ign, Sen, \text{MOD}, \models) \to (\mathbb{S}ign^\Delta, Sen^\Delta, \text{MOD}^\Delta, \models^\Delta)$

1. Each sentence of the base institution which is preserved by $\mathcal{F}$-reduced products is also modally preserved by $\mathcal{F}$-reduced products of frames.
2. Each sentence of the base institution which is preserved by $\mathcal{F}$-reduced factors is also modally preserved by $\mathcal{F}$-reduced factors of frames.
3. The set of sentences modally preserved by $\mathcal{F}$-reduced products of frames is closed under possibility $\Diamond$.
4. The set of sentences modally preserved by $\mathcal{F}$-reduced factors of frames is closed under possibility $\Diamond$.
5. The set of sentences modally preserved by $\mathcal{F}$-reduced products of frames is closed under existential χ-quantification, when χ is conservative and preserves $\mathcal{F}$-reduced products in the base institution.
6. The set of sentences modally preserved by $\mathcal{F}$-reduced factors of frames is closed under existential χ-quantification, when χ lifts $\mathcal{F}$-reduced products of frames.

7. The set of sentences modally preserved by $\mathcal{F}$-reduced factors of frames and the set of sentences modally preserved by $\mathcal{F}$-reduced products of frames are both closed under (finite) conjunction.
8. The set of sentences modally preserved by $\mathcal{F}$-reduced products of frames is closed under infinite conjunctions.
9. If a sentence is modally preserved by $\mathcal{F}$-reduced factors of frames then its negation is modally preserved by $\mathcal{F}$-reduced products of frames.

And finally, if we further assume that $\mathcal{F}$ contains only ultrafilters,

10. If a sentence is modally preserved by $\mathcal{F}$-reduced products of frames then its negation is modally preserved by $\mathcal{F}$-reduced factors of frames.
11. The set of sentences modally preserved by both $\mathcal{F}$-reduced products and factors of frames is closed under negation.

□

COROLLARY 1. Each modal sentence obtained from the Łoś-sentences of the base institution by iterative application of logical connectives, necessity □, possibility ◇, and χ-quantifications for which χ is conservative, preserves reduced products of models (in the base institution), and lifts reduced products of frames

1. is modally preserved by ultraproducts and ultrafactors, and
2. is preserved by ultraproducts.

□

Note that (ordinary) preservation by ultrafactors is not a property to be in general expected for modal satisfaction, since, unlike in the case of ultraproducts, for ultrafactors the (ordinary) preservation cannot be established as a consequence of the modal preservation. This seems to be one of the important particularities of modal satisfaction.

Similarly to the corresponding result from (Diaconescu, 2003), the only conditions that narrow the set of modal sentences which are preserved by ultraproducts refer to the quantifiers. Except lifting of reduced frames, the other conditions are at the level of the base institution and they have been previously analysed. Therefore, the key condition is the lifting of reduced frames. However the result below reduces it to lifting of models (in the base institution).

DEFINITION 6. A signature morphism $\chi\colon \Sigma \to \Sigma'$ is Φ^Δ-*exact* when the square of the naturality of β^Δ for χ is pullback:

$$
\begin{array}{ccccc}
\Sigma & & \mathrm{Mod}(\Sigma) & \xleftarrow{\beta^\Delta_\Sigma} & \mathrm{Mod}'(\Sigma\Phi^\Delta) \\
\chi \downarrow & & \mathrm{Mod}(\chi) \uparrow & & \uparrow \mathrm{Mod}(\chi\Phi^\Delta) \\
\Sigma' & & \mathrm{Mod}(\Sigma') & \xleftarrow[\beta^\Delta_{\Sigma'}]{} & \mathrm{Mod}'(\Sigma'\Phi^\Delta)
\end{array}
$$

□

PROPOSITION 2. Let $\chi\colon \Sigma \to \Sigma'$ be a signature morphism in the base institution. If χ is Φ^Δ-exact and lifts and preserves reduced products of models, then it lifts reduced products of frames. □

COROLLARY 2. A signature morphism lifts reduced products of frames when it is finitary representable conservative, Φ^Δ-exact, and preserves reduced products of models (in the base institution). □

EXAMPLE 1. We may recall that for the institution of first order logic, each injective signature morphism adding only a finite number of constants as new symbols is finitary representable. Moreover, such signature morphism is also Φ^{Δ}-exact when we take the sub-institution with signatures are sets of sort symbols plus constant declarations as the domain institution. This shows how the preservation of modal sentences by ultraproducts in conventional modal first order logic becomes a special case of Theorem 1. □

Let us derive a compactness property for modal institutions as application of our modal preservation result.

DEFINITION 7. A set of sentences E for a signature Σ is *consistent* if E^* is not empty.

An institution is *model compact* if each set of sentences is consistent when all its finite subsets are consistent.

If for each set of sentences E and each sentence e, $E \models e$ implies the existence of a finite subset $E_f \subseteq E$ such that $E_f \models e$, then we say that the institution is *compact*. □

REMARK 1. Each model compact institution having negation is compact and each compact institution having **false**[1] is model compact. □

Recall from (Diaconescu, 2003) the following result:

PROPOSITION 3. Any institution in which each sentence is preserved by ultraproducts is model compact. □

When we apply this to modal institutions we obtain:

COROLLARY 3. If the base institution is a Łoś-institution and each modal sentence is obtained from the sentences of the base institution by iterative application of logical connectives, necessity $\Box$, possibility $\Diamond$, and χ-quantifications for which χ is finitary representable, conservative, Φ^{Δ}-exact, and preserves reduced products of models (in the base institution), then the modal institution is model compact. □

EXAMPLE 2. A typical concrete instance of Corollary 3 is conventional modal first order logic, which is therefore model compact. By Remark 1 modal first order logic is compact too. □

4. Conclusions and Future Research

We have defined abstract 'modal' institutions having frames defined from the models of the base institution, modal sentences as extensions of the sentences of the base institution with the usual modal operators as sentences, and a modal satisfaction between frames and sentences. We can extent our work to the multi - modal case. Other logical systems such as epistemic logic, action logic, dynamic logic or deontic logic which use Kripke models may benefit from our approach. Based on the present work and on the Grothendieck constructions on institutions (Diaconescu, 2002) we plan to develop an institution underlying formal ethics.

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[1] A sentence which is not satisfied by any model.

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