Logics of Imperfect Information

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Abstract

The paper contains a survey of results and interpretations of incomplete information in predicate and modal logics.

1 Extensive games

It is customary to present games in classical game theory in *extensive form.* This is done by first fixing a set of actions A which represents the set of possible choices of the players in the game and then define an *extensive game* G_A of perfect information is a tuple

$$G_A = (N, H, Z, (I_i)_{i \in N}, P, (u_i)_{i \in N})$$

such that

(i) N is the set of players of the game;

(ii) H is a set of sequences of actions from A, which are called *histories*, or *plays* of the game. We require that:

(a) If $h \in H$, then any initial segment of h is in H too;

(b) There is a history h_0, the root of the game, which is an initial segment of every $h \in H$;

(iii) Z is the set of maximal histories of the game;

(iv) Each I_i is the information set of player i, which in the case of games of perfect information is a singleton.

(v) $P : H \backslash Z \rightarrow N$ is the player function which assigns to every non-terminal history the player whose turn is to move;

(vi) each u_i is the payoff function for player $i \in N$, that is, a function which specifies for each maximal history in Z what is the payoff for player i.

From the class of extensive games of perfect information, we single out a particular subclass, which is the class of z*ero-sum* (*win-loss*) games. These are games played by two players, that is, $\exists$ and $\forall$ and are defined in the standard way.

For any nonterminal history $h \in H$ we let $A(h)$ be the set of actions available to a player at the history h, i.e. $A(h) = \{x \in A : h \frown x \in H\}$, and $P^{-1}(\{i\})$ be the set of histories of H where it is player i's turn to move, as specified by the function P. A *strategy* for a player i is usually defined as any function

$$f_i : P^{-1}(\{i\}) \to A$$

such that $f_i(h) \in A(h)$. In other words, a strategy for a player in the game yields exactly one choice for any position where the player has to move.

An old result due to Zermelo is that every finite (i.e. game in which all the histories have finite length) extensive zero-sum game is determined: either $\exists$ or $\forall$ has a winning strategy in the game.

2 Strategies as plans of action

It is natural to introduce a more operative notion of strategy as a *plan of action*: only the *histories reached* in the game using the strategy matter; other counterfactual situations are ignored. We disregard the information sets of the players for the present case.

Fix a game $G_A = (\{\exists, \forall\}, H, Z, (I_i)_{i \in \{\exists,\forall\}}, P, (u_i)_{i \in \{\exists,\forall\}})$. A plan of action for a player in the game is a set of histories which contains the initial history, is closed with respect to the moves of the opponent, and yields, for every position reached by using the strategy, at least one possibility to continue the game.

More precisely, a *plan of action* for $\exists$ in the game G_A is a set $S_\exists \subseteq H$ which satisfies the following conditions:

(a) $h_0 \in S_\exists$.

For every nonterminal history h:

(b) If $h \in S_\exists$, and $P(h) = \forall$, then $h \frown x \in S_\exists$ for every $x \in A(h)$.

(c) If $h \in S_\exists$, and $P(h) = \exists$, then $h \frown x \in S_\exists$ for at least one $x \in A(h)$.

(d) The plan of action $S_\exists$ is a *winning* one, if in addition, for every maximal $h \in S_\exists \cap Z : u_\exists(h) = 1.$■

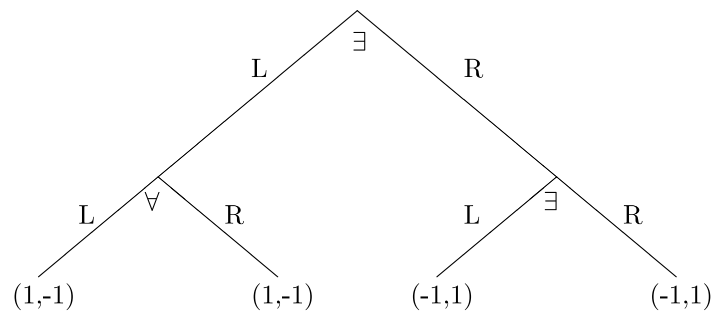
A plan of action for $\forall$ is defined in the same way, excep that '$\exists$' is replaced by '$\forall$' and in clause (d) we have: $u_\forall(h) = -1$.

A plan of action as defined above is in the general case *non-deterministic* in that it need not *functionally* pick up the choice to be made by the player in question. If we want it to be a function, then we need the following qualification. A set $S_\exists \subseteq H$ is a *deterministic plan of action* for $\exists$ if there is a function

$$f_\exists : P^{-1}(\{\exists\}) \cap S_\exists \to A$$

such that $S_\exists$ satisfies conditions (a)-(b) above and in addition condition (c) is replaced by (c') If $h \in S_\exists$, and $P(h) = \exists$, then $h \frown f_\exists(h) \in S_\exists$. A deterministic plan of action for $\forall$ is defined analoguously.

Notice that in the example below, $\exists$ has a winning strategy as a plan of action but not in the traditional sense:



The deterministic winning strategy is $f_\exists(\varnothing) = L$.

3 Semantical games of perfect information

When the set of actions A consists of the set of the subformulas $Subf(\varphi)$ of a proposition sentence φ in negation normal form (otherwise negation is treated as role swapping) and a model M of the language of φ is fixed in such a way that

- Each history $h \in H$ is a sequence of subformulas of φ;
- The root of the tree is φ;
- When the last member of a history h is $\psi \wedge \theta$ ($\psi \vee \theta$), then both $h \frown (\psi)$ and $h \frown (\theta)$ belong to H and the corresponding move is by $\forall$ ($\exists$);
- A maximal history is a win for $\exists$ if its last member is true in M; otherwise it is a win for $\forall$

then we have a semantical game $G_{\varphi;M}$ for propositional logic. Game-theoretical truth and falsity in M are defined in an obvious way:

$$
\begin{aligned}
M \vDash^{+}_{GT} \varphi &\iff \text{there is a winning plan of action for } \exists \text{ in } G_{\varphi;M}\\
M \vDash^{-}_{GT} \varphi &\iff \text{there is a winning plan of action for } \forall \text{ in } G_{\varphi;M}.
\end{aligned}
$$

A fairly simple proof shows that game theoretical truth coincide with Tarskian truth, and game-theoretical falsity coincides with Tarskian falsity.

When the formula φ is a formula of predicate logic such that the players choose not only conjunctions and disjunctions but also elements from the universe of the model corresponding to the universal and existential quantifiers of the formula, then we have a semantical game $G_{\varphi,g;M}$ where g is a partial assignment to the free variables of φ ($\varnothing$ is the empty assignment). In this case the root of the tree is (φ, g) and each nonmaximal history h is extended either with

$$(\chi, g), \chi \in \{\psi, \theta\},$$

or with

$$(\chi, g \cup \{(x_i, a)\}),\ a \in dom(M).$$

The details are straightforward.

Game-theoretical truth and game theoretical falsity are defined in analogy with the propositional case. It turns out that, if the *Axiom of Choice* is assumed, the existence of winning deterministic plans of action for $\exists$ and $\forall$ coincides with Tarskian truth and falsity, respectively.

When a formula is in prenex normal form, then any deterministic winning plan of action for $\exists$ is decomposable into *Skolem functions*; and if it is false, then any deterministic winning plan of action for player $\forall$ is decomposable into *Kreisel's counterexamples.*

4 Semantical games of imperfect information

The *extensive game* G_A of *imperfect information* is a tuple

$$G_A = (\{\exists, \forall\}, H, Z, P, (I_i)_{i \in \{\exists, \forall\}}, (u_i)_{i \in \{\exists, \forall\}})$$

where all the sets are as before, except for the information sets $(I_i)_{i \in \{\exists, \forall\}}$which are not any longer singletons. Each I_i is a partition of the set of histories where player i is to move, that is, a partition of the set $\{h \in H : P(h) = i\}$. The histories h, h' are *equivalent for player* i, $h \sim_i h'$, if there is $S \in I_i$ such that $h, h' \in S$.

There are usually two requirements on equivalent histories.

The Consistency condition (to equivalent histories there should correspond indistinguishable futures)

$$\text{For every } h, h' \in H, \text{ and } i \in N:\ h \sim_i h' \Rightarrow A(h) = A(h').$$

The *von Neumann & Morgenstern condition* (*to equivalent histories there should correspond indistinguishable pasts*)

$$\text{For every } h, h' \in H, \text{ and } i \in N : h \sim_i h' \Rightarrow length(h) = length(h')$$

A deterministic plan of action for player i is defined exactly as before, except that now the function f_i is required to be *uniform*:

$$\text{For every } h, h' \in S_i\text{: If } h \sim_i h' \Longrightarrow f_i(h) = f(h').$$

Imperfect information does at least three things:

- It introduces *indeterminacy* in the game.

- It allows for a phenomenon known in game theory as *signalling*.
- It introduces, in combination with contradictory negation, *paraconsistency* in the logic.

I shall discuss some examples in the full paper (Sandu and Pietarinen 2003).

4.1 Imperfect information in predicate logic

We consider extensions of first-order logic with formulas like

$$\forall x_0(\exists x_1/\{x_0\})\varphi$$
$$\forall x_0\exists x_1\forall x_2(\exists x_3/\{x_0,x_1\})\psi$$

where φ and ψ are standard first-order formulas. The idea is that in the extensive form of the corresponding games (played on a fixed model M), any two histories

$$\langle\langle\forall x_0(\exists x_1/\{x_0\})\varphi,\varnothing\rangle,\langle(\exists x_1/\{x_0\})\varphi,\{(x_0,a)\}\rangle\rangle$$

and

$$\langle\langle\forall x_0(\exists x_1/\{x_0\})\varphi,\varnothing\rangle,\langle(\exists x_1/\{x_0\})\varphi,\{(x_0,b)\}\rangle\rangle$$

from the first game are indistinguishable for $\exists$. In the second example, any two histories (we omit the left side formulas and the root of the tree)

$$\langle\{(x_0,a)\},\{(x_1,b)\},\{(x_2,c)\}\rangle$$

and

$$\langle\{(x_0,a')\},\{(x_1,b')\},\{(x_2,c)\}\rangle$$

are indistinguishable for $\exists$. The idea in the general case

$$Qx_0Qx_2...Qx_{n-1}(Qx_n/W)\chi$$

with $W\subseteq\{x_1,...,x_{n-1}\}$ is that the slash '/' introduces an equivalence relation on the set of histories of the relevant game and thus any two histories

$$\langle\{(x_0,a_0)\},\{(x_1,a_1)\},...,\{(x_{n-1},a_{n-1})\}\rangle$$

and

$$\langle\{(x_0,b_0)\},\{(x_1,b_1)\},...,\{(x_{n-1},b_{n-1})\}\rangle$$

are equivalent for player i exactly when

$$\text{For each } i\in(\{x_1,...,x_{n-1}\}-W): a_i=b_i.$$

The resulting logic (IF-predicate logic) is known to have the following properties:

1. **Effective disproof procedure**: The set of IF-contradictions is recursively axiomatizable.

2. **Compactness**: A set Γ of $IF-$sentences has a model if and only if every finite subset of Γ has a model.

3. **Interpolation property**. Let K_1 and K_2 be two class of structures definable by $IF-$sentences. If K_1 and K_2 are disjoint, then there is class of models K definable by a first-order sentence such that

$$K_1 \subseteq K \text{ and } K \cap K_2 = \varnothing.$$

4. **Expressive power**. Σ_1^1-logic has greater expressive power than standard first-order logic. For instance, the property of being a non-standard number, an infinite set. etc are definable by IF-sentences.

5. **Definability of truth**: $IF-$logic defines its own truth-predicate, in the sense that there is a $IF-$formula $\Phi(x)$, such that for every model M of PA, and every $IF-$sentence φ in the similarity type of PA:

$$M \models \Phi(\ulcorner\varphi\urcorner) \Leftrightarrow M \models \varphi.$$

I will discuss some of the foundational issues in the full paper (Sandu and Hyttinen 2001).

4.2 Informational independence and restricted quantifiers

This kind of informational independence arises in typically two cases.

1. With restricted quantifiers in predicate logic, that is, with quantifiers of the form '$\exists x : R(x)$' and '$\forall x : Q(x)$' where '$R(x)$' and '$Q(x)$' are relations in the relevant model. In this case we shall have formulas of the form

$$(\forall x_0 : R_0(x_0))(\exists x_1 : R(x_1)/\{x_0\})\varphi$$
$$(\forall x_0 : R_0(x_0))(\exists x_1 : R(x_1))(\forall x_2 : R(x_2))(\exists x_3 : R(x_3)/\{x_0, x_1\})\psi$$

The games are exactly like before, the players' choosing individuals from the relevant model which belong to the extension of the appropriate relation R_i (when there is no legal move, the opponent wins right away). The assumption of imperfect information is implemented, as above, as the requirement of uniformity over equivalent strategies. The *Consistency Condition* or the *von Neumann* & *Morgenstern Condition* may be violated in this case. I shall discuss few examples in the full paper.

2. With modal operators which can be regarded as restricted quantifiers over accessibility relations. In the syntax, we add to standard modal logic formulas of the form

$$Q_1Q_2...Q_{n-1}(Q_n/W)\varphi$$

where each Q_i is one of the standard modal operators $\Box$ or $\Diamond$ and $W \subseteq \{1, ..., n-1\}$. Models have the form $M = (W, R, V)$. Games now will be played starting with a possible world w_0 and the players will choose possible worlds along the accessibility relation R (the play stops right away if a player cannot continue with a legal move, with the other player winning the play right away). As in the previous case, the idea here is that any two histories

$$\langle w_0, w_1, ..., w_{n-1}\rangle$$

and

$$\langle w_0, w_1^{'}, ..., w_{n-1}^{'} \rangle$$

are equivalent for the relevant player exactly when

$$\text{For all } i \in (\{0, ..., n-1\} - W) : w_i = w_i^{'}.$$

It is known (Tulenheimo 2003) that the resulting logic (IF-modal logic) is strictly stronger than standard modal logic.

In the full paper I will discuss some other interpretations of imperfect information than the uniformity of plans of action interpretation.

References

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