COMBINING INTERPRETED LANGUAGES IN ABSTRACT ALGEBRAIC LOGIC*

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Algebraic logic can be viewed the part of logic that focuses on the logical equivalence of sentences rather than their validity. This is especially true of abstract algebraic logic where the way in which equivalence and validity interconnect is the central object of study. Algebraic methods have proved useful in several areas of logic when an abstract semantical approach is called for.

Abstract algebraic logic has developed along two distinct lines. In the *Boolean* or *semantic-based* approach, logical equivalence is taken to be a primitive notion, while in the *logistic* or *rule-based* approach it is obtained from a formal system of deduction by abstracting the classical *Lindenbaum-Tarski process*. This is a preliminary report on an ongoing project aimed at developing a general framework that combines these two approaches. Hopefully this will lead to a suitable domain for defining and investigating, in an algebraic context, fibring [15] and other forms of combining logics.[1]

We trace our understanding of the relation between equivalence and validity back to Frege's seminal insight, as interpreted by Church [6], that a (declarative) sentence is to be viewed as a proper name. In particular it *denotes* or *names* something (Church's rendering of the Frege's *bedeuten*). The thing it denotes, i.e., its *denotation* (*Bedeutung*), Church calls its *truth-value*. According to Frege a sentence also has a *sense* (*Sinn*). Carnap [5] makes a similar distinction between the *extension* and the *intension* of a sentence. However, while Carnap's notion of *extension* conforms closely to that of truth-value, for *intension* he clearly has in mind something different from Frege's sense. He uses it to abstract the property common to all sentences that are logically equivalent. We will use *extension*, *truth-value*, and *denotation* synonymously. We reserve the term *intension*, as does Carnap, for the property that abstracts the relation of logical equivalence.

Languages

The algebraization of a propositional logic is essentially a one-step process, but the algebraization of a quantifier logic requires two steps, the first being a transformation into a propositional language. When starting with a quantifier logic,

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[1]A similar, category-theoretic framework for abstract logic can be found in the theory of *institutions* [7, 10]. For a categorical approach to fibring see [13].

we assume that the transformation has already been done, so our development begins with a propositional logic and hence a propositional language. A *language type* is a pair $\langle \mathcal{C}, P \rangle$; members of $\mathcal{C}$ are called *logical connectives*, and members of P *atomic sentences*. The finite *rank*, or *arity*, of each $\omega \in \mathcal{C}$ is denoted by $\boldsymbol{\rho}\omega$. The set P of atomic sentences is called the *signature* of the language type. We assume $\mathcal{C} \cap P = \emptyset$.

Let $\langle \mathcal{C}, P \rangle$ be a language. The $(\mathcal{C}, P)$*-sentences*, P*-sentences* or simply *sentences* for short, are defined recursively in the usual way. The set of $(\mathcal{C}, P)$-sentences is denoted by $\mathrm{Se}_{\mathcal{C}}(P)$ or $\mathrm{Se}(P)$. In many contexts the set $\mathcal{C}$ of logical connectives is fixed while the set of atomic sentences varies. For example, in first-order logic the Boolean connectives, quantifiers, and equality are fixed, but the set of extralogical relation symbols often varies. By a *family of language types* we mean a possibly proper class $\big\langle \langle \mathcal{C}, P \rangle : P \subseteq \mathcal{P} \big\rangle$ languages of different signatures but with the same set $\mathcal{C}$ of logical connectives.

The $\mathcal{C}$*-sentential forms* are defined like sentences except that *sentential variables*, *variables* for short, take the place of atomic sentences. Finally, for each $P \subseteq \mathcal{P}$ the $(\mathcal{C}, P)$*-formulas*, or simply P*–formulas*, are defined like the sentential forms except that both sentential variables and atomic sentences of P are taken to be formulas. The class of sentential forms is denoted by $\mathrm{Sf}_{\mathcal{C}}$ or Sf and the class of formulas by $\mathrm{Fm}_{\mathcal{C}}(P)$ or $\mathrm{Fm}(P)$.

If φ is a sentential form, or more generally a P-formula, we write φ in the form $\varphi(v_0, \dots, v_{n-1})$ to indicate that the variables that occur in φ must appear in the list $v_0, \dots, v_{n-1}$. Then for any $\psi_0, \dots, \psi_{n-1} \in \mathrm{Se}(P)$, we denote by $\varphi(\psi_0, \dots, \psi_{n-1})$ the P-sentence (more precisely the $(\mathcal{C}, P)$-sentence) that results from φ by simultaneously substituting ψ_k for each occurrence of v_k.

For convenience, in the sequel we will also use *formula* as a generic term referring to sentences, formulas, or sentential forms. Technically, the class of formulas includes both sentences and sentential forms ($\mathrm{Fm}_{\mathcal{C}}(P) \supseteq \mathrm{Sf}_{\mathcal{C}} \cup \mathrm{Se}_{\mathcal{C}}(P)$), so no confusion is likely.

Languages as algebras

Let $\langle \mathcal{C}, P \rangle$ be an arbitrary language type. By an *algebra of language type* $\langle \mathcal{C}, P \rangle$, we mean a system $\boldsymbol{A} = \langle A, \omega^{\boldsymbol{A}}, \boldsymbol{p}^{\boldsymbol{A}} \rangle_{\omega \in \mathcal{C}, \boldsymbol{p} \in P}$, where A is a nonempty set (called the *universe* or *carrier* of $\boldsymbol{A}$), $\omega^{\boldsymbol{A}}$ is a $\boldsymbol{\rho}\omega$-ary operation on A (i.e., $\omega^{\boldsymbol{A}} \colon A^{\boldsymbol{\rho}\omega} \to A$) for each $\omega \in \mathcal{C}$, and, for each $\boldsymbol{p} \in P$, $\boldsymbol{p}^{\boldsymbol{A}}$ is a 0-ary operation of A, called a *distinguished element* of $\boldsymbol{A}$. $\mathcal{C}$ is called the *logical language type* of $\boldsymbol{A}$ and P is its *signature*. Note that the language type of $\boldsymbol{A}$ is a pair whose components are the logical language type and the signature of $\boldsymbol{A}$.

The *algebra of sentences of signature* P is $\langle \mathrm{Se}(P), \omega^{\mathbf{Se}(P)}, \boldsymbol{p} \rangle_{\omega \in \mathcal{C},\, \boldsymbol{p} \in S}$. It is denoted by $\mathbf{Se}(P)$. Corresponding to the family of languages $\big\langle\, \langle \mathcal{C}, P \rangle : P \subseteq \mathcal{P} \,\big\rangle$ we have a *family of sentence algebras* $\langle\, \mathbf{Se}(P) : P \subseteq \mathcal{P} \,\rangle$. The *algebra of sentential forms* $\mathbf{Sf} = \langle \mathrm{Sf}, \omega^{\mathbf{Sf}} \rangle_{\omega \in \mathcal{C}}$ and the *algebra of* P*-formulas*

$$\mathbf{Fm}(P) = \langle \mathrm{Fm}(P), \omega^{\mathbf{Fm}(P)}, \boldsymbol{p} \rangle_{\omega \in \mathcal{C},\, \boldsymbol{p} \in S}$$

are defined similarly. The signature of the first is empty and the signature of the second is P. We also have a *family of formula algebras* $\langle\, \mathbf{Fm}(P) : P \subseteq \mathcal{P} \,\rangle$.

Interpreted languages and the Leibniz relation

The language has been formalized; the next step is to formalize the notion of an interpretation and with it the notions of meaning, intension, and extension. A $(\mathcal{C}, P)$-*matrix* is a pair $\mathcal{A} = \langle \boldsymbol{A}, F_{\mathcal{A}} \rangle$ where $\boldsymbol{A}$ is a $(\mathcal{C}, P)$-algebra and $F_{\mathcal{A}}$ a subset of the universe A of $\boldsymbol{A}$ called the *designated filter* of $\mathcal{A}$. The sentences of a (loosely) interpreted language will take their meanings in the universe A of some $(\mathcal{C}, P)$-matrix $\mathcal{A}$. $F_{\mathcal{A}}$ is the set of meanings of true sentences (under the given interpretation), and its complement $A \setminus F_{\mathcal{A}}$ the set of meanings of the false sentences. The set of meanings inherits its algebraic structure from the sentence algebra via the compositionality of the meaning function. Thus attached to each $(\mathcal{C}, P)$-matrix $\mathcal{A} = \langle \boldsymbol{A}, F_{\mathcal{A}} \rangle$ is a canonical homomorphism from $\mathbf{Se}(P)$ to $\boldsymbol{A}$ that will carry the meanings of the sentences in the interpretation. The *nominal* elements of $\mathcal{A}$ are those that are actually the meaning (under a given interpretation) of some sentence. $\mathcal{A}$ is *nominal* if each element if nominal.

Definition 1. Let $\langle \mathcal{C}, P \rangle$ be an arbitrary language type. By an (*algebraic*) *interpretation* of $\langle \mathcal{C}, P \rangle$ we will mean any nominal $(\mathcal{C}, P)$-matrix $\mathcal{A} = \langle \boldsymbol{A}, F_{\mathcal{A}} \rangle$. The canonical surjective homomorphism from $\mathbf{Se}(P)$ onto $\boldsymbol{A}$ is called the *meaning function* or *meaning homomorphism* of $\mathcal{A}$ and is denoted by $\mathrm{mng}_{\mathcal{A}}$. $\boldsymbol{A}$ is called the *underlying meaning algebra* and $F_{\mathcal{A}}$ the *truth set* of $\mathcal{A}$.

In a context in which $\mathcal{C}$ is fixed we often refer to a $(\mathcal{C}, P)$-interpretation as a *P-interpretation*.

Definition 2. (i) By an (*algebraically*)*interpreted language* we mean a language type $\langle \mathcal{C}, P \rangle$ together with an single interpretation of $\langle \mathcal{C}, P \rangle$.

(ii) A *loosely* (*algebraically*)*interpreted language* is a language type together with a class of interpretations, called *admissible*.

(iii) A *loosely* (*algebraically*)*interpreted language family* is a language type family $\big\langle \langle \mathcal{C}, P \rangle : P \subseteq \mathcal{P} \big\rangle$ together with a class of *admissible* interpretations for each $\langle \mathcal{C}, P \rangle$, $P \subseteq \mathcal{P}$.

The consequence relation naturally associated with any loosely interpreted language has all the properties normally associated with such relations.

Definition 3. A sentence φ is said to be a *consequence* of a set of sentences Γ in a loosely interpreted language, in symbols $\Gamma \vDash \varphi$, if, for every admissible interpretation $\mathcal{A}$, $\mathrm{mng}^{\mathcal{A}}(\varphi) \in F_{\mathcal{A}}$ whenever $\mathrm{mng}^{\mathcal{A}}(\Gamma) \subseteq F_{\mathcal{A}}$.

With both the meaning and truth-value (extension) of a sentence clear in an interpreted language, we can define its intension in terms of the notion of indiscernibility of meanings in the interpretation.

Definition 4. Let $\mathcal{A} = \langle \boldsymbol{A}, F_{\mathcal{A}} \rangle$ be a nominal $(\mathcal{C}, P)$-matrix.

(i) Two elements $a, b \in A$ are said to be *$\mathcal{A}$-indiscernible* if, for every $\vartheta(v) \in \mathrm{Fm}(P)$ with a single sentential variable v, we have $\vartheta^{\boldsymbol{A}}(a) \in F_{\mathcal{A}}$ iff $\vartheta(b) \in F_{\mathcal{A}}$.

(ii) Let $\Omega_{\mathcal{A}} = \{ \langle a, b \rangle \in A^2 : a \text{ and } b \text{ are indiscernible in } \mathcal{A} \}$. $\Omega_{\mathcal{A}}$ is called the *Leibniz relation* on $\mathcal{A}$.

Theorem 5. *Let $\mathcal{A}$ be a nominal $(\mathcal{C}, P)$-matrix. Then the Leibniz relation is a congruence relation on* $\boldsymbol{A}$ *that is compatible with* $F_{\mathcal{A}}$ *in the sense that, if* $a \equiv b \pmod{\Omega_{\boldsymbol{A}}}$ *and* $a \in F_{\mathcal{A}}$*, then* $b \in F_{\mathcal{A}}$*. Moreover,* $\Omega_{\mathcal{A}}$ *is the largest congruence on* $\boldsymbol{A}$ *with this property.*

The P-sentences φ and ψ have the same *intension* in a P-interpretation $\mathcal{A}$ if their meanings are $\mathcal{A}$-indiscernible, i.e., if $\mathrm{mng}^{\mathcal{A}}(\varphi) \equiv \mathrm{mng}^{\mathcal{A}}(\psi) \pmod{\Omega_{\mathcal{A}}}$. They have the same *extension* if they have the same truth-value, i.e., either $\mathrm{mng}^{\mathcal{A}}(\varphi)$ and $\mathrm{mng}^{\mathcal{A}}(\psi)$ are both in $F_{\mathcal{A}}$ or are both in $\bar{F}_{\mathcal{A}}$, where $\bar{F}_{\mathcal{A}} = A \setminus F_{\mathcal{A}}$. Abstracting, the *intension* of a sentence φ is taken to be the equivalence class of its meaning under the Leibniz congruence, i.e., $\mathrm{mng}^{\mathcal{A}}(\varphi)/\Omega_{\mathcal{A}}$. Its *extension* is its truth-value, and is true or false depending on whether of not $\mathrm{mng}^{\mathcal{A}}(\varphi) \in F_{\mathcal{A}}$.

An interpreted language is said to be *Fregean* if the extensionality relation is a congruence. In this case the intensionality and extensionality relations coincide. A loosely interpreted language is *Fregean* if each of its interpretations is Fregean.

An interpretation $\mathcal{A}$ is *intensional* if the meaning of every sentence is uniquely determined by its intension, i.e., if $\Omega_{\mathcal{A}}$ is the identity relation. In an intensional interpretation the meaning of any sentence can be identified with its intension. Extending this terminology we say that a (loosely) interpreted language is *intensional* if its unique interpretation is intensional (all its interpretations are intensional). By the *intensional reduction* of an interpretation $\mathcal{A}$ we mean the quotient matrix $\mathcal{A}^* = \langle \boldsymbol{A}/\Omega_{\mathcal{A}}, F_{\mathcal{A}}/\Omega_{\mathcal{A}} \rangle$.

Theorem 6. *An intentional interpreted language is Fregean iff its unique interpretation is a two-element matrix.*

Definition 7. Two sentences φ and ψ in a loosely interpreted language are said to be *logically equivalent* if φ and ψ have the same intension, i.e., $\mathrm{mng}^{\mathcal{A}}(\varphi) \equiv \mathrm{mng}^{\mathcal{A}}(\psi) \pmod{\Omega_{\mathcal{A}}}$, in every interpretation $\mathcal{A}$.

Since the intensionality and extensionality relations coincide for Fregean interpreted languages, we have that in a Fregean loosely interpreted langauge, two sentences are logically equivalent iff they have the same truth-value in every interpretation. So as one would expect, our notion of logical equivalence coincides with Carnap's for Fregean loosely interpreted languages.

Two sentences are logically equivalent in a loosely interpreted language iff they are logically equivalent in its intensional reduction. This follows easily from the fact that the Leibniz congruence is compatible with the truth set.

Rule-based interpreted languages

Rule-based interpretations are derived directly from the deductive mechanism of the logic. Let $\mathcal{C}$ be a set of logical connectives. A *rule over* $\mathcal{C}$ is an ordered pair of the form $\langle \Gamma, \varphi \rangle$ where $\Gamma \subseteq \mathrm{Sf}_{\mathcal{C}}$ and $\varphi \in \mathrm{Sf}_{\mathcal{C}}$. Its *cardinality* is the cardinality of its set of premisses, i.e., $|\Gamma|$. A rule of cardinality 0 is called an axiom and identified with its conclusion φ.

Definition 8. A *rule-based logic* over a logical language type $\mathcal{C}$ is a triple $\mathcal{D} = \langle \mathcal{C}, \mathrm{Ax}_{\mathcal{D}}, \mathrm{Ru}_{\mathcal{D}} \rangle$, where $\mathrm{Ax}_{\mathcal{D}}$ is a set of axioms and $\mathrm{Ru}_{\mathcal{D}}$ a set of rules over $\mathcal{C}$. The *cardinality* of $\mathcal{D}$ is the smallest cardinal greater than the cardinality of each rule in $\mathrm{Ru}_{\mathcal{D}}$.

An interpretation of a rule-based logic is obtained by adjoining extra-logical axioms to the logical axioms and rules. Let $\mathcal{D} = \langle \mathcal{C}, \mathrm{Ax}_{\mathcal{D}}, \mathrm{Ru}_{\mathcal{D}} \rangle$ be a rule-based logic and P a signature. A set T of P-sentences (more precisely $(\mathcal{C}, P)$-sentences) is called a *$\mathcal{D}$-theory over P* if it contains all substitution instances of the axioms and is closed under the inference rules of $\mathcal{D}$. T is *axiomatized* by a set Γ of sentences if it is the smallest theory including Γ.

Definition 9. Let $\mathcal{D}$ be a rule-based logic over $\mathcal{C}$. An interpretation of a language type $\langle \mathcal{C}, P \rangle$ is *$\mathcal{D}$-based* if it is the intensional reduction of a matrix of the form $\langle \mathbf{Se}(P), T \rangle$ for some theory $\mathcal{D}$-theory T over P. It is said to be *axiomatized* by a set of sentences if T axiomatized by Γ.

An interpretation is *rule-based* if it is $\mathcal{D}$-based for some rule-based logic $\mathcal{D}$ over $\mathcal{C}$.

Every rule-based logic $\mathcal{D}$ over a logical language type $\mathcal{C}$ defines an intensional loosely interpreted family of languages whose admissible interpretations are the $\mathcal{D}$-based interpretations of $\langle \mathcal{C}, P \rangle$ for every signature P. A sentence φ is said to be a *$\mathcal{D}$-consequence* of a set of sentences Γ, in symbols $\Gamma \vDash_{\mathcal{D}} \varphi$, if $\mathrm{mng}_{\mathcal{A}}(\varphi) \in F_{\mathcal{A}}$ where $\mathcal{A}$ is the $\mathcal{D}$-based interpretation axiomatized by Γ.

Theorem 10. *Let $\mathcal{D}$ be rule-based logic over $\mathcal{C}$. The following are equivalent for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Se}(P)$.*

(i) $\Gamma \vDash_{\mathcal{D}} \varphi$.

(ii) *φ is a consequence of Γ in the loosely interpreted language defined by $\mathcal{D}$.*

(iii) *φ derivable from Γ by the logical axioms and rules of $\mathcal{D}$.*

First-order predicate logic gives rise to a rule-based loosely interpreted language by eliminating individual variables and formalizing substitution for individual variables in the object language. The transformed language will have as its logical language type $\mathcal{C} = \{\to, \vee, \wedge, \neg, \top, \bot\} \cup \{\, \Diamond_i : i < \omega \,\} \cup \{\, \mathsf{d}_{ij} : i, j < \omega \,\}$, where the $\Diamond_i$ are modal connectives and the d_{ij} are constants representing respectively the (transforms of the) quantifiers $\exists x_i$ and the atomic equality formulas $x_i = x_j$. The signature P contains an atomic formula r for each predicate symbol R.[2]

Semantic-based interpreted languages

semantic-based interpretations are derived from the primitive notion of *class of models*—an arbitrary class—together with the primitive notion of *meaning*, a function that assigns to each element of the class and each sentence a "meaning". The other primitive notion involved in defining interpretations is the *validity relation*. This specifies which sentences are "true" in each model. The fundamental condition that connects meaning and validity is the following: if a sentence is true in a given model, then so is any sentence with the same meaning.

Definition 11. Let $\langle \mathcal{C}, P \rangle$ be a language type. Let $\mathcal{S} = \Big\langle \langle \mathcal{C}, P \rangle, \mathsf{Mod}_{\mathcal{S}}, \mathrm{mng}_{\mathcal{S}}, \vDash_{\mathcal{S}} \Big\rangle$ be an ordered four-tuple where

[2]For details of the formalization of predicate logic as a rule-based interpreted language see [4, 11]. For a formulation of predicate logic where substitution is simulated in the object language see [12]. For a discussion of quantifiers as modal operators see [3, 14].

(i) $\mathsf{Mod}_{\mathcal{S}}$ is a class, called the *class of models* of $\mathcal{S}$,

(ii) $\mathrm{mng}_{\mathcal{S}}$ is a function with domain $\mathsf{Mod}_{\mathcal{S}} \times \mathrm{Se}(P)$, called the *meaning function* of $\mathcal{S}$, and

(iii) $\vDash_{\mathcal{S}}$ is a subclass of $\mathsf{Mod}_{\mathcal{S}} \times \mathrm{Se}(P)$, called the *validity relation* of $\mathcal{S}$.

We write $\mathfrak{M} \vDash_{\mathcal{S}} \varphi$ in place of $\langle \mathfrak{M}, \varphi \rangle \in \vDash_{\mathcal{S}}$ and $\mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\varphi)$ in place of $\mathrm{mng}_{\mathcal{S}}(\mathfrak{M}, \varphi)$.

$\mathcal{S}$ is called a *pre-semantical system* if the *conceptuality condition* holds, i.e.,

(iv) $\mathfrak{M} \vDash_{\mathcal{S}} \varphi$ and $\mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\varphi) = \mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\psi)$ implies $\mathfrak{M} \vDash_{\mathcal{S}} \psi$, for all $\mathfrak{M} \in \mathsf{Mod}_{\mathcal{S}}$ and $\varphi, \psi \in \mathrm{Se}(P)$.

$\mathcal{S}$ is called a *semantical system* if in addition the meaning function satisfies the *compositionality condition for meanings*, i.e.,

(v) For all $\mathfrak{M} \in \mathsf{Mod}_{\mathcal{S}}$, $\omega \in \mathcal{C}$, and $\varphi_0, \dots, \varphi_{\boldsymbol{\rho}\omega-1}, \psi_0, \dots, \psi_{\boldsymbol{\rho}\omega-1} \in \mathrm{Se}(P)$, if $\mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\varphi_i) = \mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\psi_i)$ for all $i < \boldsymbol{\rho}\omega$, then $\mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\omega\varphi_0 \cdots \varphi_{\boldsymbol{\rho}\omega-1}) = \mathrm{mng}^{\mathfrak{M}}_{\mathcal{S}}(\omega\psi_0 \cdots \psi_{\boldsymbol{\rho}\omega-1})$.

The validity relation $\vDash_{\mathcal{S}}$ defines a Galois connection between models and sentences in the usual way. For each set Γ of sentences define

$$\mathsf{Mo}\,\Gamma = \{\, \mathfrak{M} \in \mathsf{Mod} : \text{ for every } \varphi \in \Gamma,\ \mathfrak{M} \vDash_{\mathcal{S}} \varphi \,\},$$

and for each class K of models define

$$\mathrm{Th}\,\mathsf{K} = \{\, \varphi \in \mathrm{Se}(P) : \text{ for every } \mathfrak{M} \in \mathsf{K},\ \mathfrak{M} \vDash_{\mathcal{S}} \varphi \,\}.$$

$\mathsf{Mo}\,\Gamma$ is called the *class of models* of Γ and $\mathrm{Th}\,\mathsf{K}$ the *theory* of K. A sentence φ is said to be an $\mathcal{S}$*-consequence* of Γ, in symbols $\Gamma \vDash_{\mathcal{S}} \varphi$, if $\varphi \in \mathrm{Th}\,\mathsf{Mo}\,\Gamma$.

Let $\mathcal{S}$ be a semantical system, and let $\mathfrak{M}$ be a model of $\mathcal{S}$. We denote the range of the function $\mathrm{mng}^{\mathfrak{M}}$ for any $\mathfrak{M}$ by $\mathrm{Me}\,\mathfrak{M}$, called the *meaning set of* $\mathfrak{M}$. The structure of $\mathfrak{M}$ is embodied in its meaning set. The compositionality condition guarantees that $\mathrm{Me}\,\mathfrak{M}$ can be given the structure of a (unique) algebra.

Definition 12. Let $\mathcal{S}$ be a semantical system over the language type $\langle \mathcal{C}, P \rangle$, and let $\mathfrak{M} \in \mathsf{Mod}_{\mathcal{S}}$. By the *meaning algebra of* $\mathfrak{M}$ we mean the algebra of type $\langle \mathcal{C}, P \rangle$

$$\mathbf{Me}\,\mathfrak{M} = \langle \mathrm{Me}\,\mathfrak{M}, \omega^{\mathbf{Me}\,\mathfrak{M}}, \boldsymbol{p}^{\mathbf{Me}\,\mathfrak{M}} \rangle_{\omega \in \mathcal{C},\, \boldsymbol{p} \in P},$$

where $\omega^{\mathbf{Me}\,\mathfrak{M}}\big(\mathrm{mng}^{\mathfrak{M}}(\varphi_0), \dots, \mathrm{mng}^{\mathfrak{M}}(\varphi_{\boldsymbol{\rho}\omega-1})\big) = mng^{\mathfrak{M}}(\omega\varphi_0 \cdots \varphi_{\boldsymbol{\rho}\omega-1})$ for each $\omega \in \mathcal{C}$ and all $\varphi_0, \dots, \varphi_{\boldsymbol{\rho}\omega-1} \in \mathrm{Se}(P)$, and $\boldsymbol{p}^{\mathbf{Me}\,\mathfrak{M}} = \mathrm{mng}^{\mathfrak{M}}(\boldsymbol{p})$ for each $\boldsymbol{p} \in P$.

The meaning homomorphism $\mathrm{mng}^{\mathfrak{M}} : \mathbf{Se}(P) \twoheadrightarrow \mathbf{Me}\,\mathfrak{M}$ is unique since $\mathbf{Se}(P)$ is a minimal algebra. It is also surjective. Hence $\mathbf{Me}\,\mathfrak{M}$ is also minimal.

In addition to the structure of the meaning algebra, the semantical system also specifices the sentences "true" in $\mathfrak{M}$ by means of the validity relation. Recall that *theory of* $\mathfrak{M}$ is defined to be the set $\mathrm{Th}\,\mathfrak{M} = \{\, \varphi \in \mathrm{Se}(P) : \mathfrak{M} \vDash \varphi \,\}$. Abstracting from first-order predicate logic we get the following definition of $\mathcal{S}$-elementary equivalence.

Definition 13. Let $\mathcal{S}$ be a pre-semantical system over the language $\langle \mathcal{C}, P \rangle$. Let $\mathfrak{M}$ and $\mathfrak{N}$ be models of $\mathcal{S}$. $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{S}$*-elementarily equivalent*, in symbols $\mathfrak{M} \equiv_{\mathcal{S}} \mathfrak{N}$, if they have the same theory, i.e., $\mathrm{Th}\,\mathfrak{M} = \mathrm{Th}\,\mathfrak{N}$.

The *truth filter* of $\mathfrak{M}$, in symbols $F_{\mathfrak{M}}$, is the set of meanings of sentences in the theory of $\mathfrak{M}$, i.e., $F_{\mathfrak{M}} = (\mathrm{mng}^{\mathfrak{M}})(\mathrm{Th}\,\mathfrak{M}) \subseteq \mathrm{Me}\,\mathfrak{M}$. We combine these two data to obtain the *meaning matrix of* $\mathfrak{M}$: $\mathcal{ME}\ \mathfrak{M} = \langle \mathbf{Me}\,\mathfrak{M}, F_{\mathfrak{M}} \rangle$.

Every semantical system $\mathcal{S}$ defines a generally loosely interpreted language whose admissible interpretations are the meaning matrices of $\mathcal{S}$.

Theorem 14. *Let $\mathcal{S}$ be a semantical system over the language type $\langle \mathcal{C}, P \rangle$. For all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Se}(P)$, $\Gamma \vDash_{\mathcal{S}} \varphi$ iff φ is a consequence of Γ in the loosely interpreted language defined by $\mathcal{S}$.*

The admissible interpretations associated with a semantical system are not in general intensional, in contrast to the case for rule-based logics. Important information can be lost in passing from a meaning matrix to its intensional reduction.[3]

Two meaning matrices $\mathcal{ME}\ \mathfrak{M}$ and $\mathcal{ME}\ \mathfrak{N}$ are *isomorphic* if there is an isomorphism h: $\mathbf{Me}\,\mathfrak{M} \cong \mathbf{Me}\,\mathfrak{N}$ of the underlying algebras that preserves truth filters in the sense that $h(F_{\mathfrak{M}}) = F_{\mathfrak{N}}$. The meaning matrix captures essentially all the abstract structure of $\mathfrak{M}$ inherent in the semantical system in the sense that, if the meaning matrices of two models $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathcal{S}$ are isomorphic, then $\mathfrak{M}$ and $\mathfrak{N}$ must be $\mathcal{S}$-elementarily equivalent. To see this we simply observe that, if h is a isomorphism between the meaning matrices $\mathcal{ME}\ \mathfrak{M}$ and $\mathcal{ME}\ \mathfrak{N}$, then for every $\varphi \in \mathrm{Se}(P)$, $\varphi \in \mathrm{Th}\,\mathfrak{M}$ iff $\mathrm{mng}^{\mathfrak{M}}(\varphi) \in F_{\mathfrak{M}}$ iff $h\big(\mathrm{mng}^{\mathfrak{M}}(\varphi)\big) \in F_{\mathfrak{N}}$ iff $\mathrm{mng}^{\mathfrak{N}}(\varphi) \in F_{\mathfrak{N}}$ (by the initiality of $\mathbf{Se}(P)$) iff $\varphi \in \mathrm{Th}\,\mathfrak{N}$. The converse does not hold in general. For this purpose a further abstraction is required, namely passage to the intensional reduction. The following algebraic characterization of $\mathcal{S}$-elementary equivalence is one of the main results of the elementary theory of abstract algebraic logic.

Theorem 15. *Let $\mathcal{S}$ be a semantical system over the language type $\langle \mathcal{C}, P \rangle$. Then two models $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathcal{S}$ are $\mathcal{S}$-elementarily equivalent iff their reduced meaning matrices are isomorphic, i.e., there exists a (unique) isomorphism h: $\mathbf{Me}^*\,\mathfrak{M} \cong \mathbf{Me}^*\,\mathfrak{N}$ of the reduced meaning algebras such that $h^\star F^*_{\mathfrak{M}} = F^*_{\mathfrak{N}}$.*

Classical propositional logic, first-order predicate logic, and modal logic are just some of the logical system that can be formalized in a natural way as both rule-based and semantic-based interpreted languages. The modal logic is an example of a special class of semantic-based interpreted languages that arises as a abstraction of Kripke's "possible world" semantics.

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