Properties of Intuitionistic Provability and Preservativity Logics

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1 Introduction

In this paper we study some intuitionistic modal logics that arise from a specific mathematical interpretation of the modal operations:

$\Box\varphi$	"φ is provable in HA", i.e. $\mathsf{HA} \vdash \varphi$
$\varphi \rhd \psi$	"for all $\sigma \in \Sigma_1$: $\mathsf{HA} \vdash \sigma \to \varphi$ implies $\mathsf{HA} \vdash \sigma \to \psi$",

where HA is Heyting Arithmetic, the constructive counterpart of PA, and Σ_1 is the first level of the arithmetical hierarchy. All the logics we consider are part of *the provability or preservativity logic of* HA, the set of propositional schemes that HA proves about its provability predicate $\Box_{\mathsf{HA}}$ or its preservativity predicate $\rhd_{\mathsf{HA}}$. Provability logic, in the language $L_\Box$, can be considered to be part of preservativity logic, in the language $L_\rhd$, as $\Box A$ can be defined as $\top \rhd A$. Preservativity logic was introduced by [8] as a constructive alternative for interpretability logic. No axiomatization is known for the preservativity logic of HA, but over the last few years at least part (all?) of the logic has been axiomatized. In this paper we consider the following principles of the preservativity logic of HA ($\rhd$ and $\Box$ bind stronger than $\wedge, \vee$, that bind stronger than $\to$).

IPC	intuitionistic propositional logic		
$P1$	$A \rhd B \wedge B \rhd C \to A \rhd C$		
$P2$	$A \rhd B \wedge A \rhd C \to A \rhd (B \wedge C)$		
Dp	$A \rhd B \to (A \vee C) \rhd (B \vee C)$		
Mp	$A \rhd B \to (\Box C \to A) \rhd (\Box C \to B)$		
		K	$\Box(A \to B) \to (\Box A \to \Box B)$
$4p$	$A \rhd \Box A$	4	$\Box A \to \Box\Box A$
Lp	$(\Box A \to A) \rhd A$	L	$\Box(\Box A \to A) \to \Box A$
		Le	$\Box(A \vee B) \to \Box(A \vee \Box B)$
Rules:			
$Pres$	$A \to B \;/\; A \rhd B$	Nec	$A \;/\; \Box A$
MP	$A \;\; (A \to B) \;/\; B$		

iP^- denotes the logic given by IPC, the principles $P1, P2$, and the rules $Pres$ and MP. iP is the logic iP^- extended by Dp and is called the *basic preservativity logic* (it is complete for the natural frames, Theorem 4). By $iP4$ we denote the logic iP extended by the principle $4p$. Similarly for the other preservativity principles Lp, Mp, Wp. iK denotes the logic given by IPC, K, and the rules Nec and MP. The logic iK extended by the principle 4 is denoted by $iK4$. Conform tradition, iKL is denoted by iL, Similarly for Le; $iLLe$ denotes iL extended by Le. iPX denotes an arbitrary extension of iP. Lemma 1 below shows that all provability principles can be derived from the preservativity principles.

The non-logical axioms $K, 4, L$ are part of the provability logic GL of PA. This in contrast to Dp and Le, that do not belong to the preservativity logic of PA. The modal study of the principles above is interesting for two main reasons. First, these principles express principles of HA. Therefore, knowledge about them is likely to provide insights in HA, and might help in the search for a complete axiomatization of the provability and preservativity logic of HA. Second, as mentioned above, some of these principles do not belong to the logics regularly studied in intuitionistic modal logic and might therefore be a valuable addition to the field.

In [5] modal completeness results were presented for all logics given by some or all of these principles, except for iPL. In this paper we investigate the relation between the preservativity and provability logics (Section 3), and present fixed point theorems for both iPL and iL (Section 4). From the latter it follows that both the fixed point theorem and the Beth property hold for any extension of these logics in the appropriate language, in particular for the provability and preservativity logic of HA.

1.1 $\Box$-fragments

The $\Box$-fragment of a preservativity logic iPX in $L_\rhd$ is defined to be

$$iPX_\Box := \{A \text{ in } L_\Box \mid iPX \vdash A\}.$$

Here we ask ourselves what the $\Box$-fragment of a given preservativity logic is. An obvious relation between $\Box$ and $\rhd$ is given by the following lemma ([5]).

Lemma 1 *$iP^- \vdash \Box(A \to B) \to A \rhd B$ and $iP^- \vdash A \rhd B \to (\Box A \to \Box B)$.*

The guiding idea behind the description of the $\Box$-fragments is the translation $^\circ$ on formulas that inductively replaces all occurrences of $A \rhd B$ by $\Box A \to \Box B$. All preservativity principles except Dp, Mp are derivable in iL under this translation ([5],[4]). It turns out that there are rules that cover the effect of Dp and Mp on the $\Box$-fragment of the preservativity logics that contain them:

$$\begin{array}{ll} DR & \Box A \to \Box B \ / \ \Box(A \vee C) \to \Box(B \vee C) \\ MoR & \Box A \to \Box B \ / \ \Box(\Box C \to A) \to \Box(\Box C \to B). \end{array}$$

We show that for all preservativity logics considered in this paper, these rules determine the $\Box$-fragment of a preservativity logic in the following way.

Theorem 2 (*Numbers indicate the sections where the equality is proved.*)

$$\begin{array}{lclcl} iP_\Box & \stackrel{3.3}{=} & iK & \stackrel{3.3}{=} & iK+DR \\ iP4_\Box & \stackrel{3.1}{=} & iLe & \stackrel{3.1.1}{=} & iK4+DR \\ iPL_\Box & \stackrel{3.2}{=} & iLLe & \stackrel{3.2}{=} & iL+DR \\ iPM_\Box & \stackrel{3.3}{=} & iK & \stackrel{3.3}{=} & iK+DR+MoR \end{array}$$

In particular, if X is one of $4p, Lp$ or empty, then $iPX_\Box = iKX^\circ + DR$. For $X = Mp$, $iPX_\Box = iK + DR + MoR = iK$.

2 Semantics for Preservativity and Modal Logic

Definition 3 A frame F *is a triple* $\langle W, R, \leq\rangle$*, where* W *is a nonempty set of possible worlds, points or nodes,* $\leq$ *is a partial order and* R *is a binary relation satisfying* $(\leq \circ R) \subseteq R$.
A model M *is a quadruple* $\langle W, R, \leq, \Vdash\rangle$ *where* $\langle W, R, \leq\rangle$ *is a frame and* $\Vdash$ *is a forcing relation between points in* W *and propositional letters which satisfies the following condition, if* $x \Vdash p$ *and* $x \leq y$*, then* $y \Vdash p$*.* (persistence)

The forcing relation extends to the connectives in IPC in the usual manner:
$M, w \Vdash A \wedge B \equiv_{def} M, w \Vdash A$ and $M, w \Vdash B$;
$M, w \Vdash A \vee B \equiv_{def} M, w \Vdash A$ or $M, w \Vdash B$;
$M, w \Vdash A \to B \equiv_{def} \forall v \geq w(M, v \Vdash A$ implies $M, v \Vdash B)$;
$M, w \Vdash \top$ for any w; $M, w \nVdash \bot$ for any w.
and to $\rhd$-formulas as follows:
$M, w \Vdash A \rhd B \equiv_{def}$ for any v such that wRv, if $M, v \Vdash A$, then $M, v \Vdash B$.
Then $M, w \Vdash \Box A$ iff for any v such that wRv, $M, v \Vdash A$.
Also one has persistence for all formulas.
As a matter of fact, given the persistence for propositional letters, the condition that $(\leq \circ R) \subseteq R$ is a *necessary and sufficient* condition to guarantee persistence for all formulas ([9]), which is different from the condition $(\leq \circ R) \subseteq (R \circ \leq)$ for intuitionistic modal logic (sometimes we write $R \circ \leq$ as $\bar{R}$).

Theorem 4 ([4]) $iP \vdash A$ iff A is valid on all (finite) frames.

It turns out that all intuitionistic *modal* logics iT that we will consider below are complete with respect to some class of frames satisfying additionally:

- (*brilliancy*) $(R \circ \leq) \subseteq R$.

In particular, iK is complete w.r.t the class of finite brilliant frames.

Next we give some basic propositions in preservativity logics. First the connection between the preservation rule and the more-often-used rule: necessitation.

Theorem 5 *In any preservativity logic iT containing all theorems in iP^-, the preservation rule and the necessitation rule are equivalent.*

The following substitution lemmas are used in our section on fixed points.

Lemma 6 (a) $T \vdash \boxdot(A \leftrightarrow B) \rightarrow (F[A/p] \leftrightarrow F[B/p])$, for $T = iP4$ or $T = iK4$,
(b) If p occurs only *modalized* in F, i.e. only under $\Box$ or $\rhd$, then
$T \vdash \Box(A \leftrightarrow B) \rightarrow (F[A/p] \leftrightarrow F[B/p])$, for $T = iP4$ or $T = iK4$.

PROOF. We can prove (a) directly by induction on the complexity of F, and (b) by induction from (a). $\dashv$

As in classical provability logic 4 is derivable from L (and here also from Le).

Lemma 7 *For $T = iL$ or $T = iLe$, $T \vdash \Box A \rightarrow \Box\Box A$.*

3 Conservation Results

The rule BP (BoxPres): $\Box A \rightarrow \Box B / A \rhd B$ plays a dominant role in the following considerations. The rule is discussed in Section 5.2 of [4] where a short proof sketch is given for the admissibility of the rule for iPH. The admissibility of this rule is not automatically preserved for sublogics or superlogics. Each logic needs its own proof, but we will just give the basic one for $iP4$.

3.1 Conservation of $iP4$ over iLe

In the part of this subsection before subsubsection 3.1.1 we will repeat the argument of [4] because it is very characteristioc. A notational convention: Given a frame M, $[z] := \{w|$ there is a sequence of $w_0 S_0 w_1 \cdots w_n = w$ for some worlds $w_0, w_1, \cdots, w_n$ in M where $S_i \in \{R, \leq\}\}$. Thus $[z]$ stands for the subframe generated by z. The same notation applies to models.

Theorem 8 *1. In $L_\Box$, 4 corresponds to semi-transitivity: $(R \circ R) \subseteq (R \circ \leq)$.*
2. In $L_\Box$, $\vdash_{iK4} A$ iff A is valid on all finite transitive frames.
3. The principle $4p$ corresponds to gatheringness: if $wRvRu$, then $v \leq u$.
4. $\vdash_{iP4} A$ iff A is valid on all finite gathering frames.
5. On finite frames Le corresponds to the Le-property:
$\forall wv(wRv \rightarrow \exists x(wRx \leq v \wedge \forall u(vRu \rightarrow x \leq u)))$.
6. $\vdash_{iLe} A$ iff A is valid on all finite brilliant Le-frames.
7. In $L_\Box$, $\vdash_{iLLe} A$ iff A is valid on all finite transitive conversely well-founded brilliant Le-frames.

Lemma 9 *Let $M := \langle W, R, \leq, \Vdash\rangle$ and $N := \langle W, R', \leq, \Vdash\rangle$ be two finite models. If $R' \subseteq R \subseteq (R' \circ \leq)$, then $M, w \Vdash B$ iff $N, w \Vdash B$ for any formula B in $L_\Box$ and any world $w \in W$.*

Lemma 10 *Let $M := \langle W, R, \leq, \Vdash\rangle$ be a finite Le brilliant model. Then there is a finite gathering model $N = \langle W, R', \leq, \Vdash\rangle$ such that $R' \subseteq R \subseteq (R' \circ \leq)$.*

PROOF. Assume that $M := \langle W, R, \leq, \Vdash\rangle$ is a finite Le brilliant model. Define:

$$wR'v \equiv_{def} wRv \text{ and } \forall u(vRu \to v \leq u) \text{ and } N := \langle W, R', \leq, \Vdash\rangle.$$

This model can be shown to have the right properties. $\dashv$

Theorem 11 *$\vdash_{iLe} A$ iff A is valid on all finite gathering frames.*

PROOF. The right-to-left direction follows from the fact that Le is derivable in $iP4$ (observe that, by $4p$ and Dp, $\vdash_{iP4} (A \vee B) \rhd (A \vee \Box B)$, and apply Lemma 1). We just need to show the other direction. Suppose that $\nvdash_{iLe} A$. Then by the completeness of iLe, we know that there is a world b in some finite brilliant Le model $M = \langle W, R, \leq, \Vdash\rangle$ such that $M, b \nVdash A$. According to Lemma 10, there is another new finite gathering model $N = \langle W, R', \leq, \Vdash\rangle$ such that $R' \subseteq R \subseteq (R' \circ \leq)$. From Lemma 9, it follows that $N, b \nVdash A$. $\dashv$

Corollary 12 *(Conservation) $\vdash_{iP4} A$ iff $\vdash_{iLe} A$, for all A in $L_{\Box}$.*

3.1.1 iLe is equivalent to the logic $iK4$ with DR

Lemma 13 *Let M be a model on a gathering frame and x, y be two worlds in this model such that xRy. If $y \Vdash A$, then, for any $z \in [y)$, $y \leq z$ and $z \Vdash \boxdot A$.*

Lemma 14 *$iP4$ satisfies BP: $\vdash_{iP4} A \rhd B$ iff $\vdash_{iP4} (\Box A \to \Box B)$.*

PROOF. The direction from left to right follows from Lemma 1. We prove the other direction by contraposition. Suppose that $iP4 \nvdash A \rhd B$. It follows that $A \rhd B$ is false at a point w of some finite gathering model M. Then there is a point v such that wRv, $v \Vdash A$ and $v \nVdash B$. Take $W' := \{w\} \cup [v)$, $R' = R\restriction_{W'}$, $\leq' = \leq\restriction_{W'}$, and $x \Vdash p$ iff $x \Vdash' p$ for any propositional variable p, for all $x \in W'$. Observe that M' has a gathering frame. Note that, for any $x \in [v)$ and for any formula B in $L_{\rhd}$, $M', x \Vdash B$ iff $M, x \Vdash B$.

It is clear that $M', w \nVdash \Box B$ because $wR'v$ and $M', v \nVdash B$. By the above lemma, we get that $M', w \Vdash \Box A$ because $R'[w] \subseteq [v)$ and for any $x \in [v), x \Vdash A$. So $M', w \Vdash \Box A$ but $M', w \nVdash \Box B$, which implies that $M', w \nVdash \Box A \to \Box B$. Therefore $\nvdash_{iP4} \Box A \to \Box B$. $\dashv$

Lemma 15 *If the rule BP is admissible for iPX, then DR is admissible for iPX, and whence for $iPX_{\Box}$. If in addition $iPX \vdash Mp$, then both DR and MoR are admissible for iPX, and whence for $iPX_{\Box}$.*

Theorem 16 *iLe is equivalent to the logic $iK4$ with the extra rule DR. Whence $iP4_{\Box} = iLe = iK4 + DR$.*

PROOF. First show that iLe is contained in $iK4 + DR$. We only need to show that Le is derivable in the latter logic. Since $iK4 + DR \vdash \Box A \rightarrow \Box\Box A$, we can get Le immediately by just applying DR. For the other direction, recall Lemma 7 that the principle 4 is derivable in iLe. Whence it remains to show that DR is admissible for iLe, which is the same as showing that it is admissible for $iP4_\Box$, by Corollary 12. That DR is admissible for $iP4_\Box$ follows from the previous lemma, by applying Lemma 15. $\dashv$

3.2 Conservation of iPL

Lemma 17 *The principle Lp corresponds to gatheringness plus converse well-foundedness of the modal relation. Similarly, L corresponds to semi-transitivity plus well-foundedness* ([4]).

Extending the argument of lemma 10 we obtain:

Lemma 18 $iLLe \vdash A$ *iff* A *is valid on all finite gathering conversely well-founded frames.*

Theorem 19 (*Conservation*) *iLLe is the* $L_\Box$*-fragment of* iPL.

By a syntactic argument one gets

Lemma 20 1. $iP4 \vdash \Box((\Box C \rightarrow C) \rhd C) \rightarrow (\Box C \rightarrow C) \rhd C$.

2. $iP4 \vdash \Box \boxdot L \leftrightarrow \boxdot L \leftrightarrow \Box L$ *where* L *is* $(\Box C \rightarrow C) \rhd C$.

The following lemma is then just a matter of careful checking.

Lemma 21 (*Detour Lemma*) $iPL \vdash A$ *iff there exist* $C_1, C_2, \cdots, C_n$ *such that* $iP4 \vdash \Box((\Box C_1 \rightarrow C_1) \rhd C_1) \wedge \cdots \wedge \Box((\Box C_n \rightarrow C_n) \rhd C_n) \rightarrow A$.

Lemma 14 extends to

Lemma 22 *If* $iP4 \vdash \Box C \rightarrow (\Box A \rightarrow \Box B)$, *then* $iP4 \vdash \Box C \rightarrow (A \rhd B)$, *for all formulas* C.

Theorem 23 *iPL satisfies BP:* $iPL \vdash \Box A \rightarrow \Box B$ *iff* $iPL \vdash A \rhd B$

Corollary 24 *iLLe is equivalent to the logic iL with the extra rule DR.*
Whence $iPL_\Box = iLLe = iL + DR$.

3.3 Conservation of iPM over iK

There is an interesting but complicated model-theoretic proof that iK is the $L_\Box$-fragment of iPM (Theorem 27), again via showing the admissibility of BP. Here we skip this and just give a direct syntactic proof.
We give a proof of $iK + DR + MoR = iPM_\Box$ that uses the following translation on formulas which is related to the translation $^\circ$ given in the introduction.

Definition 25 *The translation * from formulas in $L_{\triangleright}$ to those in $L_{\Box}$ is inductively defined as follows:*

- *For p, $\top$ and $\bot$, $p^* = p$, $\top^* = \top$ and $\bot^* = \bot$.*
- *For $\circ \in \{\vee, \wedge, \rightarrow\}$, $(A \circ B)^* = A^* \circ B^*$.*
- $(\neg A)^* = \neg A^*$
- $(A \triangleright B)^* = \Box(A^* \rightarrow B^*)$.

Lemma 26 *If $iK \vdash X^*$, then $iPX_{\Box} = iK$, where X is in $L_{\triangleright}$.*

PROOF. Clearly, $iK \subseteq iPX_{\Box}$. Thus it remains to show that $iPX_{\Box} \subseteq iK$. Assume that $iPX_{\Box} \vdash A$. Of course we can consider A as a formula in $L_{\triangleright}$ according to the fact that $\Box A \equiv (\top \triangleright A)$ in iP. It suffices to show that

$$\text{if } iPX \vdash A, \text{ then } iK \vdash A^* \ (*)$$

because, for any formula B in $L_{\Box}$, $B^* = B$.
Since $iPX \vdash A$, there is a finite sequence $s_1 s_2 \cdots s_n (= A)$ of formulas in $L_{\triangleright}$ in which, for any $s_i (1 \leq i \leq n)$,

1. either s_i is in the forms of P_1, P_2, Dp or X,
2. or there are some $A_1, A_2, s_j \in L_{\Box}$ $(j < i)$ such that $s_i = A_1 \triangleright A_2$ and $s_j = A_1 \rightarrow A_2$,
3. or there are some $s_j, s_k (j, k < i)$ such that $s_k = s_j \rightarrow s_i$.

The sequence $s_1^* s_2^* \cdots s_n^* (= A^*)$ of formulas in $L_{\Box}$ is a proof of A^* in iK. We treat the first case and leave the others to the reader. If s_i is an instance of P_1, P_2, Dp or X, then it is easy to see that s_i^* is a theorem of iK for the first three, and it follows by assumption for X.

$\dashv$

Theorem 27 $iPM_{\Box} = iK = iK + DR + MoR$.

Theorem 28 $iP_{\Box} = iK$.

4 Fixed Points and Beth Definability

In this section we will give the fixed point theorems for iL and iPL and point out connections with Beth's Definability Theorem. Let us remind the reader that fixed point theorems are of the form: for each formula $A(p)$ in which p occurs only modalized, there exists a unique B not containing p such that B and $A(B)$ are provably equivalent. The proof of the existence of fixed points in iL is an adaptation of the well-known proof of that property for GL; the proof of the existence of fixed points in iPL derives from the one for IL, the basic interpretability logic ([2]). The connections with Beth's theorem extend the work of [1] on interpretability logic (see also [3], Ch. 5).

A notational convention: AB is the result of substitution of B for p in the formula Ap.

Theorem 29 (*Uniqueness Theorem*) *Suppose that p occurs modalized in A, then $\vdash_L (\boxdot(p \leftrightarrow Ap) \wedge \boxdot(q \leftrightarrow Aq)) \rightarrow (p \leftrightarrow q)$ where $L \in \{iL, iPL\}$.*

The proof of the uniquenss theorem in e.g. [7] is intuitionistically acceptable. Proofs of the existence of fixed points for a system usually consist of proving the existence of fixed points for the basic formulas and proving an inductive step. For the inductive step for iPL, we may borrow the following (reformulated) Theorem 2.4 of [2]), since its proof did not use classical logic. This means that for iL and iPL we can confine ourselves to proving the basic cases.

Theorem 30 *Let U be any extension of iL or iPL satisfying:*

> *FIX: Every formula Ap of the form $\Box Bp$ or $Bp \rhd Cp$ has a fixed point.*

and the substitution lemmas (Lemma 6). Then, for every formula Ap with p modalized, there is a formula J such that p does not occur in J and $\vdash_U J \leftrightarrow AJ$.

4.1 Fixed Point Theorems for iL and iPL

Lemma 31 *$iL \vdash \Box A\top \leftrightarrow \Box A\Box A\top$ for all formulas A.*

Now an application of Theorem 30 suffices.

Theorem 32 *If in C the propositional letter p occurs exclusively under $\Box$, then there is a formula D not containing p such that $iL \vdash D \leftrightarrow CD$.*

The following theorem is proved similarly to the one for interpretability logic in [2]. To put it more precisely, the fixed point for the formula $A(p) \rhd B(p)$ in iPL is a mirror image of that for the formula $A(p) \rhd_i B(p)$ in IL. This is not surprising since classically $A(p) \rhd B(p)$ is equivalent to $\neg B(p) \rhd_i \neg A(p)$ in IL.

Theorem 33 (*Fixed Point Theorem for $A(p) \rhd B(p)$*) *$\vdash A\Box B\top \rhd B\top \leftrightarrow A(A\Box B\top \rhd B\top) \rhd B(A\Box B\top \rhd B\top)$.*

Since we have now proved FIX of Theorem 30 we can conclude

Theorem 34 (*Fixed Point Theorem*) *For every formula Ap with p modalized, there is formula J such that p does not occur in J and $\vdash_{iPL} J \leftrightarrow AJ$.*

In iPW, we have a simpler form of fixed point for $Ap \rhd Bp$.

Theorem 35 *In iPW, the fixed point of $Ap \rhd Bp$ is $A\top \rhd B\top$.*

4.2 Beth Definability and Fixed Points

For a large class of intuitionistic modal logics the Beth property (Definition 36) and the fixed point property (Definition 37) are equivalent. This theorem and its proof is an adaptation of the corresponding theorems and proofs of [1] concerning interpretability logic. An essential difference lies in a change in Maximova's trick ([6]) needed for intuitionistic logic to obtain the Beth property from the existence of fixed points.

Definition 36 (*Beth Definability Property*) *A logic $\mathcal{L}$ has* the Beth Property *iff for all formulas $A(\bar{p}, r)$ the following holds:*

- *If $\vdash_L \boxdot A(\bar{p}, r) \wedge \boxdot A(\bar{p}, r') \to (r \leftrightarrow r')$, then there exists a formula $C(\bar{p})$ such that $\vdash_L \boxdot A(\bar{p}, r) \to (C(\bar{p}) \leftrightarrow r)$.*

Definition 37 (*Fixed Point Property*) *A logic $\mathcal{L}$ has the* fixed point property *iff, for any formula $A(\bar{p}, r)$ which is modalized in r, there exists a formula $F(\bar{p})$ such that*

- (*existence*) $\vdash_L F(\bar{p}) \leftrightarrow A(\bar{p}, F(\bar{p}))$
- (*uniqueness*) $\vdash_L \boxdot(r \leftrightarrow A(\bar{p}, r)) \wedge \boxdot(r' \leftrightarrow A(\bar{p}, r')) \to (r \leftrightarrow r')$.

Theorem 38 (*Equivalence of Beth Definability and Fixed Points*) *Let $\mathcal{L}$ be an intuitionistic logic with modal operators that extends iL and obeys the substition lemmas. Then $\mathcal{L}$ satisfies the Beth theorem iff $\mathcal{L}$ has the fixed point property.*

Since iL and iPL have fixed points, so do their extensions. Hence

Corollary 39 *Let $\mathcal{T}$ be an extension of iL or iPL satisfying the conditions of the above theorem. Then $\mathcal{T}$ has the Beth property.*

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