Formalizing Concurrent Common Knowledge as Product of Modal Logics

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Abstract

This paper introduces a two-dimensional modal logic to represent agents' *concurrent common knowledge* in distributed systems. Unlike common knowledge, concurrent common knowledge is a kind of agreement reachable in asynchronous environments. As a proper semantics to concurrent common knowledge we present the *closed sub-product of modal logics*. We axiomatize the presented logic issuing an idea of the soundness and completeness proofs.

1 Introduction

The common knowledge is a present phenomenon in a lot of situations in our social life. To coordinate actions, to establish agreements and in other typical behaviors, the individuals need a previous knowledge or the mutual understanding or even the common knowledge of certain facts. The knowledge about the conventions among all the members in a community is an example of common knowledge, once, for every stipulated fact, everybody knows this fact, and everybody knows that everybody knows such fact, and everybody knows that everybody knows that everybody knows the fact, and so on. In Computer Science, the analysis and the applications of the common knowledge and other knowledge types became a very active research field, especially in the last two decades, giving rise to the epistemic logics or logics of knowledge. However, it is proved in [8] that common knowledge requires coordinated actions and simultaneity to be attained. Hence, common knowledge can not be achieved in asynchronous systems, because simultaneity is not applicable in such environments.

We propose a logic to represent other concepts of knowledge that can be achieved in asynchronous environments, such as the *concurrent common knowledge* [12]. To illustrate the concept of concurrent common knowledge, suppose that we are attending the final game of the Soccer World Championship and our country is one of the teams. We can suppose there is a small gap of time in the arrival of the images in televisions around the country, in other words, suppose that the images reach first some places and some time later other places. As soon as the victory goal happens, some places begin to celebrate the title, knowing that in all the country, sooner or later, everybody will know about the victory. In this case, the knowledge about the winner team is not simultaneous, but everybody knows that, in some moment sooner or later, all the others will know it. Thus, we say that the team's victory is concurrent common knowledge among all.

2 A Model for Asynchronous Distributed Systems

Consider a model for asynchronous distributed systems based on Lamport's definitions of time and causality [10]: time is given by causality relations among events and consistent global states are *consistent cuts* in an asynchronous run hypergraph. The model consists in: a network of *fifo* channels with m agents; a set R of asynchronous runs; a set E of events; a set C of consistent cuts.

The hypergraph in Figure 1 illustrates one possible run of the PIF (propagation of information with feedback) algorithm for 3 agents. The goal of PIF algorithm is to make the message $\mathcal{M}$ known to all the agents in the system, and, assuming that just one agent initiates the algorithm, to inform the initiator when $\mathcal{M}$ has reached all of them.

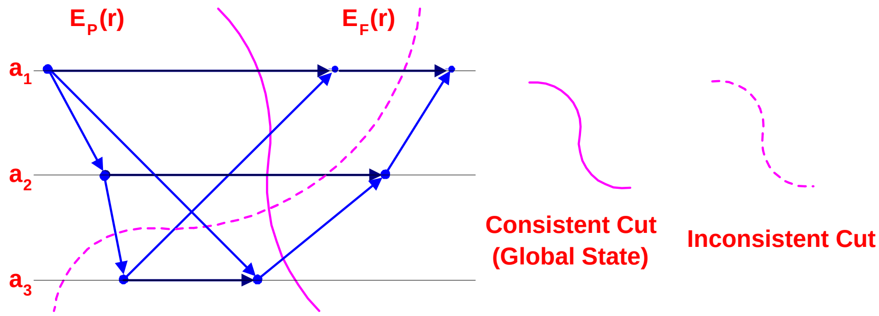

Figure 1: Consistent Cut

The dots represent **events** - when an agent sends and/or receives messages. The arrows establish a causality relation among events. A cut represents a global state and divides the graph into two sets of events, E_P and E_F, those which happen before (in the past of) and those which happen after (in the future of) the present cut. Intuitively, we can think about a **consistent cut** as a global state in which there are no messages from the future to the past.

In this model, an agent can not distinguish between two cuts if his local state is the same in both cuts. If so, the cuts are said to be *indistinguishable* according to the agent's point of view. There are distinct possible runs depending on the order in which messages reach the agents.

3 Products of Modal Logics

We think about asynchronous systems as a two-dimensional world. That is, taking into account the model of the previous section, we reason about the agents' knowledge under the perspective of a cartesian pair (r, c), a *run-cut* pair. In a modal logic approach, that means the interpretation of possible worlds are pairs (r, c) representing a state: a consistent cut c in an asynchronous run r.

The two-dimensional approach of knowledge can be formalized using the concept of *products of modal logics*. Like fibring, fusion, splitting and temporalization, which are forms to compose or decompose logics, products of modal logics is a technique to combine logics giving rise to many-dimensional or multidimensional logics. In multidimensional logics, the states are tuples representing dimensions where logical formulas are evaluated. The foundations of multidimensional logics are in Segerberg [13]. A complete overview on this subject and many results, including transference results between the whole and the component logics, were recently published in [6]. It follows some formal definitions and results on axiomatizing products of modal logics [14].

Definition 3.1 *Let $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ be two propositional frames. The* product of frames *[14] is the frame $F_1 \times F_2 = (W_1 \times W_2, R_h, R_v)$, where:*
$R_h = \{((x, z), (y, z)) | xR_1y\}$ *and* $R_v = \{((z, x), (z, y)) | xR_2y\}$.
Let L_1 and L_2 be modal logics, $\mathbf{F}(L_1)$ the class of frames validating L_1 and $\mathbf{F}(L_2)$ the class of frames validating L_2. The product of logics L_1 and L_2 is the logic $L_1 \times L_2 = \mathbf{L}(\mathbf{F}(L_1) \times \mathbf{F}(L_2))$ [1].

Definition 3.2 *For L_1 n-modal and L_2 m-modal logics, let $[L_1, L_2] = L_1 * L_2 + C^1_{ij} + C^2_{ij}$, where:*
$L_1 * L_2$ is the fusion of L_1 and L_2, $C^1_{ij} = (\Box_i \Box_{j+n} p \leftrightarrow \Box_{j+n} \Box_i p)$ and $C^2_{ij} = (\Diamond_i \Box_{j+n} p \rightarrow \Box_{j+n} \Diamond_i p)$, for $1 \le i \le n$, $1 \le j \le m$.
We say the logics L_1, L_2 are commutative *if $L_1 \times L_2 = [L_1, L_2]$.*

Definition 3.3 *A modal formula is pseudo-transitive if it has the form:*
$\bigtriangledown_1 \Box_k p \rightarrow \triangle_2 p$, where $p \in Prop$, $\bigtriangledown_1 = \Diamond_i, \ldots, \Diamond_j$, $\triangle_2 = \Box_i, \ldots, \Box_j$ are sequences of modal operators.
A PTC formula is a pseudo-transitive or closed formula.
A PTC logic is a modal logic axiomatized by PTC formulas.

Theorem 3.1 *The logic resulting from the product of two PTC modal logics is* commutative *[14]. That is, if L_1 and L_2 are PTC then $L_1 \times L_2 = [L_1, L_2]$.*

[1] The modal logic $\mathbf{L}(\mathbf{F})$ for a class of frames $\mathbf{F}$ is defined as the intersection $\bigcap\{\mathbf{L}(F) \mid F \in \mathbf{F}\}$.

Many known modal logics are PTC, such as $\mathcal{D}, \mathcal{K}4, \mathcal{S}4, \mathcal{T}, \mathcal{B}, \mathcal{S}5$, and others. Thus, two-dimensional products like $\mathcal{T} \times \mathcal{T}$, $\mathcal{S}4 \times \mathcal{S}4$ or $\mathcal{S}5 \times \mathcal{S}5$ are *commutative*. We are interested, particularly, in the commutative product $\mathcal{S}5_m \times \mathcal{S}5_m$ which is used when axiomatizing our logic.

4 Semantics for Two-Dimensional Modal Logic of Knowledge

To model the desired two-dimensional knowledge approach we need a two-dimensional many-modal logic. The two dimensions refer to the runs' and cuts' dimensions, represented by the modal operators H_i and V_i, respectively. As usual, the semantics is based on Kripke's semantics of possible worlds, so we have possibility or accessibility relations for each dimension. These accessibility relations are equivalence relations, reflecting the concept of indistinguishable cuts and runs according to the agent's point of view. Hence, the accessibility relations are, in fact, equivalence relations for indistinguishability in each level of knowledge considered: the run dimension, the cut dimension, and a third relation for the transitive closure under the former. The closure relation gives us new features, for instance, representing knowledge properties according to indistinguishable pairs (r, c). Thus, the modal operator K_i related to the closure relation $\sim_i$represents the so-called agent's *concurrent knowledge*, that is, what he knows under indistinguishable consistent cuts in all possible runs.

We introduce the definition of *closed sub-product of modal logics* in order to formalize the kind of knowledge that we are interested in. The closed sub-product of modal logics is similar to the product, with two additional features: an extra relation for the transitive closure under the two basic relations, and a subset W_Δ of the cartesian product $R \times C$. To understand the set W_Δ, consider that there are some pairs (r, c) which, in fact, may not occur in the system. If so, we restrict the evaluation of the formulas to the so-called *reasonable* pairs, that is, the pairs (r, c) that really make sense. The subset $W_\Delta \subseteq R \times C$ denote these reasonable pairs. The idea is to make the modal operators H_i and V_i range only over the reasonable pairs in W_Δ, whereas the operators $\overline{H}_i$ and $\overline{V}_i$ range over the whole cartesian product $W = R \times C$.

Definition 4.1 *Closed Sub-product of Modal Logics.*

Let L_H be the smallest set of formulas containing the set of primitives $Prop_H$, closed under negation, conjunction and the modal operators H_i, $i = 1, \ldots, m$. Let L_V be the smallest set of formulas containing the set of primitives $Prop_V$, closed under negation, conjunction and the modal operators H_i, $i = 1, \ldots, m$.

Consider the frames $F_H = (R, \cong_i)$ and $F_V = (C, \asymp_j)$ for L_H and L_V, respectively. The closed sub-product *of the frames F_H and F_V related to W_Δ is the frame $F_H \otimes F_V = (W, \simeq_i, \approx_j, \sim_k, W_\Delta)$, where:*

1. *$W = R \times C$ is the set of all the states (r, c);*
2. *$W_\Delta \subseteq W = R \times C$ is a subset of the states (r, c);*
3. *$\simeq_i = \{((r, c), (r', c)) | r \cong_i r'\}$;*
4. *$\approx_j = \{((r, c), (r, c')) | c \asymp_j c'\}$;*
5. *$\sim_k = (\simeq_i \cup \approx_j)^*$*, where $(\simeq_i \cup \approx_j)^*$ is the transitive closure under the union of $\simeq_i$ and $\approx_j$.

Let $\mathbf{F}(L_H)$ the class of frames validating L_H and $\mathbf{F}(L_V)$ the class of frames validating L_V. The *semantic sub-product* of the logics L_H and L_V is the logic $\mathbf{L}(\mathbf{F}(L_H) \otimes \mathbf{F}(L_V))$.

Definition 4.2 *Model for Closed Sub-product of Modal Logics.*

A model *M over a closed sub-product frame $F = F_H \otimes F_V$ is a pair $M = (F, v)$, where v is a truth-value function for the primitive $Prop = Prop_H \cup Prop_V$. For each $p \in Prop$, $v(p)$ is the set of (r, c) where p is true, that is, $v(p) : Prop \rightarrow 2^{R \times C}$.*

According to [12], to incorporate *concurrent common knowledge*, we need more three modalities:

. $P_i\alpha$ meaning "there is another consistent cut in the *same run* indistinguishable under the point of view of agent i where α is true". In our logic, the operator P_i is, in fact, the dual of V_i.

. $E_C\alpha$ meaning "everybody concurrently knows α", which is given by the formula $E_C\alpha = \bigwedge K_i P_i \alpha$.

. $C_C\alpha$ meaning "α is concurrent common knowledge". As usual, concurrent common knowledge implies that everybody concurrently knows α and everybody concurrently knows that everybody concurrently knows α and so on. Thus, $C_C\alpha$ is given by the formula $C_C\alpha \rightarrow E_C\alpha \wedge E_C^2\alpha \wedge E_C^3\alpha \wedge \ldots$.

We will use the same subscript i for the relations and modal operators, because we have m agents, and therefore, the product of two m-modal logics. It follows the formal semantics definitions.

Definition 4.3 *Satisfiability in L_m^2.*

Let L_m^2 be the smallest set of formulas containing Δ, the set of primitives $Prop = Prop_H \cup Prop_V$, closed under negation, conjunction and the modal operators $\overline{H}_i$, $\overline{V}_i$, K_i, E_C and C_C where $i = 1, \ldots, m$.

Suppose $\simeq_i$, $\approx_i$ and $\sim_i$ are equivalence relations in a closed sub-product of two modal frames, as defined in 4.1. Let $F = (W, \simeq_i, \approx_i, \sim_i, W_\Delta)$ be a frame for L^2_m and let M be a model over F. A formula $\alpha \in L^2_m$ is true in $[M,(r,c)]$, $[M,(r,c)] \models \alpha$, for $(r,c) \in W = R \times C$, when: [2]

1. $[M,(r,c)] \models p \Leftrightarrow (r,c) \in v(p)$, *where* $p \in Prop$;
2. $[M,(r,c)] \models \alpha \wedge \beta \Leftrightarrow [M,(r,c)] \models \alpha$ *and* $[M,(r,c))] \models \beta$;
3. $[M,(r,c)] \models \neg\alpha \Leftrightarrow [M,(r,c)] \not\models \alpha$;
4. $[M,(r,c)] \models \overline{H}_i\alpha \Leftrightarrow \forall(r',c')\{((r,c) \simeq_i (r',c')) \Rightarrow [M,(r',c')] \models \alpha\}$;
5. $[M,(r,c)] \models \overline{V}_i\alpha \Leftrightarrow \forall(r',c')\{((r,c) \approx_i (r',c')) \Rightarrow [M,(r',c')] \models \alpha\}$;
6. $[M,(r,c)] \models \Delta \Leftrightarrow (r,c) \in W_\Delta \subseteq W = R \times C$;
7. $[M,(r,c)] \models H_i\alpha \Leftrightarrow [M,(r,c)] \models \Delta$ *and* $[M,(r,c)] \models \overline{H}_i\alpha$;
8. $[M,(r,c)] \models V_i\alpha \Leftrightarrow [M,(r,c)] \models \Delta$ *and* $[M,(r,c)] \models \overline{V}_i\alpha$;
9. $[M,(r,c)] \models Q_i\alpha \Leftrightarrow [M,(r,c)] \models H_i\alpha$ *and* $[M,(r,c)] \models V_i\alpha$;
10. $[M,(r,c)] \models K_i\alpha \Leftrightarrow [M,(r,c)] \models \Delta$ *and* $\forall(r',c')\{((r,c) \sim_i (r',c')) \Rightarrow [M,(r',c')] \models \alpha\}$;
11. $[M,(r,c)] \models P_i\alpha \Leftrightarrow [M,(r,c)] \models \neg V_i \neg\alpha$;
12. $[M,(r,c)] \models E_C\alpha \Leftrightarrow [M,(r,c)] \models \bigwedge K_iP_i\alpha$;
13. $[M,(r,c)] \models C_C\alpha \Leftrightarrow [M,(r,c)] \models E^k_C\alpha$ *for all* $k \geq 1$.

5 Axiomatic System $\mathcal{C}^2_m$

When the accessibility relations are equivalence relations, we know that logics as L_H and L_V are axiomatized by $\mathcal{S}5_m$ [8]. Furthermore, we know that the product $\mathcal{S}5_m \times \mathcal{S}5_m$ is commutative, and therefore, axiomatized by $[\mathcal{S}5_m, \mathcal{S}5_m]$, according to the theorem 3.1. Thus, we propose the system $\mathcal{C}^2_m$ in the table 1 as an axiomatization of the two-dimensional many-modal logic L^2_m.

The axioms 1 to 12 are axioms of $\mathcal{S}5_m$ for the horizontal and vertical dimensions, and also for the concurrent knowledge K_i. The axioms 13, 14 and 15 reflect the properties of the commutative product. The axioms 16 and 17 restrict the knowledge to the reasonable pairs. The axioms for concurrent knowledge K_i are 18, 19 and 20. The operator Q_i defined in axiom 18 is an auxiliary one which we call "inter-dimensional step". Remembering that P_i is the dual of V_i, axioms 21 and 22 define E_C, everybody concurrently knows. And finally, for concurrent common knowledge C_C, we have axiom 23.

Table 1: **System $\mathcal{C}^2_m$**

Axioms

1. $(\overline{H}_i\alpha \wedge \overline{H}_i(\alpha \to \beta)) \to \overline{H}_i\beta$	12. $\neg K_i\alpha \to K_i\neg K_i\alpha$
2. $\overline{H}_i\alpha \to \alpha$	13. $\overline{H}_i\overline{V}_j\alpha \leftrightarrow \overline{V}_j\overline{H}_i\alpha$
3. $\overline{H}_i\alpha \to \overline{H}_i\overline{H}_i\alpha$	14. $\neg\overline{H}_i\neg\overline{V}_j\alpha \to \overline{V}_j\neg\overline{H}_i\neg\alpha$
4. $\neg\overline{H}_i\alpha \to \overline{H}_i\neg\overline{H}_i\alpha$	15. $\neg\overline{V}_i\neg\overline{H}_j\alpha \to \overline{H}_j\neg\overline{V}_i\neg\alpha$
5. $(\overline{V}_i\alpha \wedge \overline{V}_i(\alpha \to \beta)) \to \overline{V}_i\beta$	16. $H_i\alpha \leftrightarrow \Delta \wedge \overline{H}_i\alpha$
6. $\overline{V}_i\alpha \to \alpha$	17. $V_i\alpha \leftrightarrow \Delta \wedge \overline{V}_i\alpha$
7. $\overline{V}_i\alpha \to \overline{V}_i\overline{V}_i\alpha$	18. $Q_i\alpha \leftrightarrow H_i\alpha \wedge V_i\alpha$
8. $\neg\overline{V}_i\alpha \to \overline{V}_i\neg\overline{V}_i\alpha$	19. $K_i\alpha \leftrightarrow Q_iK_i\alpha$
9. $(K_i\alpha \wedge K_i(\alpha \to \beta)) \to K_i\beta$	20. $K_i(\alpha \to Q_i\alpha) \to (\alpha \to K_i\alpha)$
10. $K_i\alpha \to \alpha$	21. $P_i\alpha \leftrightarrow \neg V_i\neg\alpha$
11. $K_i\alpha \to K_iK_i\alpha$	22. $E_C\alpha \leftrightarrow \bigwedge K_iP_i\alpha$
	23. $C_C\alpha \leftrightarrow E_C(\alpha \wedge C_C\alpha)$

Rules

R0. From $\vdash \alpha$ infer every uniform substitution for α
R1. From $\vdash \alpha, \alpha \to \beta$ infer β (*modus ponens*)
R2. From $\vdash \alpha$ infer $\overline{H}_i\alpha$ (*horizontal generalization*)
R3. From $\vdash \alpha$ infer $\overline{V}_i\alpha$ (*vertical generalization*)
R4. From $\vdash \alpha$ infer $K_i\alpha$ (*two-dimensional generalization*)
R5. From $\vdash \alpha \to E_C(\alpha \wedge \beta)$ infer $\alpha \to C_C\beta$ (*induction rule*)

where $i, j = 1, \ldots, m$.

Note: Axiom 10 can be deduced from axioms 19 and 20.

[2] Because of clarity we keep the semantic definitions for abbreviations such as $P_i\alpha$ and $E_C\alpha$

6 Conclusions

This work presents results on epistemic logics and many-dimensional logics with applications in the area of distributed multi-agent systems. We introduced the axiomatic system C_m^2 for concurrent common knowledge. The system is suitable to represent the properties of concurrent knowledge in distributed systems because the semantics is based on a model which considers consistent cuts and asynchronous runs to define time.

We have used the concept of multidimensional logics to deal with the two-dimensional approach of knowledge. Thus, the main contributions of this paper is in combination of logics in order to express the desired properties and the interactions among all the involved entities. The closed sub-product of modal logics was defined to make the necessary adjustments, resulting in a more powerful semantics.

As future developments, we would like to build a temporal version of the two-dimensional knowledge logic, which would better describe the evolution of knowledge acquisition over time.

A Soundness and Completeness for $\mathcal{C}_m^2$

A.1 Soundness for $\mathcal{C}_m^2$

We prove soundness for system $\mathcal{C}_m^2$ with respect to the class of closed sub-product of modal frames for $\mathcal{S}5_m$. Thus, is it necessary to show that all the axioms of the system $\mathcal{C}_m^2$ are valid in this class of frames and the inference rules also preserve the validity. Let $\mathbf{F}$ be the class of closed sub-product of modal frames according to definition 4.1 and let M be a model over $F \in \mathbf{F}$. As $\simeq_i$, $\approx_i$ and $\sim_i$ are equivalence relations, the axioms 1 to 4 as well as the axioms 5 to 8 and 9 to 12 correspond to axioms from $\mathcal{S}5_m$, therefore the proofs can be found in [8]. As 13, 14, 15 are the axioms of Shehtman and Gabbay for the commutative product of logics, the soundness and completeness proofs can be found in [14]. For axioms 16, 17, 18, 21 and 22 soundness is straightforward from semantics rules 7, 8, 9, 11 and 12, respectively, in the satisfiability definition 4.3. The proofs for axioms 19 and 20 are not difficult and can be found in [4]. For the soundness proof of axiom 23, we propose a graph-theoretical characterization for concurrent common knowledge, as follows.

Definition A.1 *K_i-reachable, V_i-reachable, K_iV_i-reachable and KV-reachable in n KV-steps.*
Let W_Δ be the set of states (r, c) such that $M, (r, c) \models \Delta$. Let $w, w', w'', w_n \in W_\Delta, n \geq 0$.
1. w' is K_i-reachable from w if and only if $w \sim_i w'$;
2. w' is V_i-reachable from w if and only if $w \approx_i w'$;
3. w' is K_iV_i-reachable from w if and only if there is a w'' such that $w \sim_i w''$ and $w'' \approx_i w'$;
4. w' is KV-reachable from w in n KV-steps if and only if there are $w_0, w_1, \ldots, w_n$ such that $w = w_0$, $w' = w_n$, and for all j, $0 \leq j \leq n-1$ we have w_{j+1} is K_iV_i-reachable from w_j, $i \in \{1, 2, \ldots, m\}$.

Proposition A.1 *Graph-theoretical Characterization for Concurrent Common Knowledge.*
a) $M, w \models E_C\alpha$ iff, for $i \in \{1, 2, \ldots, m\}$, for all w' K_iV_i-reachable from w, $M, w' \models \alpha$;
b) $M, w \models E_C^j\alpha$ iff, for $i \in \{1, 2, \ldots, m\}$, for all w' KV-reachable from w in j KV-steps, $M, w' \models \alpha$;
c) $M, w \models C_C\alpha$ iff, for $i \in \{1, 2, \ldots, m\}$, for all w' KV-reachable from w in n KV-steps, for all $n > 0$, $M, w' \models \alpha$.

Proof for proposition A.1: part *a*) follows from definition of E_C, everybody concurrently knows; considering that $M, w \models E_C^{j+1}\alpha \Leftrightarrow M, w \models E_C(E_C^j\alpha)$, part *b*) follows by induction on k; part *c*) is straightforward from *b*) and from the definition of concurrent common knowledge C_C.

Using proposition A.1, it is easy to prove soundness for axiom 23: $M \models C_C\alpha \leftrightarrow E_C(\alpha \wedge C_C\alpha)$. For instance, for the direction $(\rightarrow)$, suppose that $M, w \models C_C\alpha$. Thus, $M, w' \models \alpha$ for all w' KV-reachable from w in n KV-steps, $n > 0$. Particularly, if w'' is KV-reachable from w in one KV-step, we have $M, w'' \models \alpha$ and $M, w' \models \alpha$ for all w' KV-reachable from w'' in n KV-steps, $n > 0$. Therefore, $M, w'' \models \alpha \wedge C_C\alpha$ for all w'' KV-reachable from w in one KV-step. Hence, $M, w \models E_C(\alpha \wedge C_C\alpha)$. The proof for the converse direction is similar.

Regarding the inference rules of $\mathcal{C}_m^2$, it is easy to see that rules R1 to R4 preserve validity, and the proofs can be found in [4]. To prove soundness for rule R5, that is, to prove that if $M \models \alpha \rightarrow E_C(\alpha \wedge \beta)$, then $M \models \alpha \rightarrow C_C\beta$, we also use the graph-theoretical characterization of proposition A.1. In fact, we show by induction on n, that for all w' KV-reachable from w in n KV-steps, $n > 0$, we have $M, w' \models \alpha \wedge \beta$, and, therefore, $M, w \models C_C(\alpha \wedge \beta)$. The complete proof is found in [4].

A.2 Completeness for $\mathcal{C}_m^2$

We prove completeness for $\mathcal{C}_m^2$ with respect to the class $\mathbf{F}$ of closed sub-product frames. Thus, it is necessary to show that every valid formula in the class $\mathbf{F}$ is a theorem from $\mathcal{C}_m^2$. Or, equivalently, we have to prove that for every formula φ $\mathcal{C}_m^2$-consistent there is a model based on a frame $F \in \mathbf{F}$ that satisfies φ. In [4] we build such finite models, that is, we prove that the system has the f.m.p. property, and therefore, as $\mathcal{C}_m^2$ is a finite axiomatization, we have, in addition, decidability. We also prove that the frames of such models are indeed frames in the class of closed sub-product frames $\mathbf{F}$.

The proof is standard, that is, the finite model is based on a frame $F^\varphi = (W^\varphi, \simeq_i^\varphi, \approx_i^\varphi, \sim_i^*, W_\Delta^\varphi)$ where $\varphi \in L_m^2$, W^φ contains all the φ-maximal $\mathcal{C}_m^2$-consistent sets, and the relations are defined as usual. The Truth Lemma is proved by induction on the length of the sub-formulas $\alpha \in Sub(\varphi)$. Consider, for instance, that we want to prove $(M^\varphi, w) \models C_C\alpha \Rightarrow C_C\alpha \in w$. Suppose that $(M^\varphi, w) \models C_C\alpha$. As $w \in W^\varphi$ is φ-maximal $\mathcal{C}_m^2$-consistent, the conjunction of the sub-formulas in w is also a finite formula in L_m^2. Let $\hat{w}$ be the conjunction of the formulas in w. Consider the set $U = \{u \in W^\varphi | (M^\varphi, u) \models C_C\alpha\}$ of states which satisfy $C_C\alpha$. Let γ be the disjunction of such states, $\gamma = \bigvee_{u \in U} \hat{u}$. As U is finite, then γ is a formula of L_m^2 and can be considered as the formula which characterizes the states where $C_C\alpha$ is true. Note that $\gamma \rightarrow E_C(\alpha \wedge \gamma)$ is $\mathcal{C}_m^2$-consistent and, therefore, we have $\vdash \gamma \rightarrow E_C(\alpha \wedge \gamma)$. By the induction rule R5, we have $\vdash \gamma \rightarrow C_C\alpha$. As $w \in U$, then $\vdash \hat{w} \rightarrow \gamma$, and thus $\vdash \hat{w} \rightarrow C_C\alpha$ (*). Hence, $C_C\alpha \in w$, otherwise $\neg C_C\alpha$ together with (*) would make w $\mathcal{C}_m^2$-inconsistent. The whole completeness proof, including the proof for the converse, is found in [4].

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