FIBRING ALGEBRAIZABLE CONSEQUENCE SYSTEMS [1]

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Abstract

In this article we study the process of (categorial) fibring of consequence systems which are algebraizable in the sense of Blok and Pigozzi. We investigate the question of preservation of algebraizability by fibring, proving that, under certain assumptions, we can preserve algebraizability by fibring of algebraizable consequence systems (with and without sharing connectives). We also study the particular case of algebraizable consequence systems which have Hilbert-style axiomatics. Finally, based on the work of Jánossy *et alia* we construct a category of equivalent algebraic semantics, proving that it is isomorphic to the subcategory of the category of algebraizable consequence systems in which there exists fibring. This suggest the possibility of fibring algebraic semantics.

Introduction

Since the introduction of the concept of Fibring of Logics by D. Gabbay in the 90's (cf. [5]), such a name was used to define a very broad set of methodologies. With the purpose of formalize the techniques of fibring, Sernadas *et alia* introduced a categorial definition of fibring, (see, for instance, [8, 11, 3]). One of the main issues intrinsic to the idea of fibring is to find preservation results. That is, the problem to know whether certain properties of the logic systems are preserved through the fibring. In the present article, based on these techniques, we study the question of *preservation of algebraizability* by fibring.

As an answer to this question we will offer some conditions for the preservation of algebrizability via fibring. Thus, we will define the category of algebraizable consequence systems ALCO, which will be our framework wherein the problem of fibring will be analyzed. Then, we will obtain the principal preservation result: In an special subcategory of ALCO (called ALCO*) it is possible to obtain the fibring of algebraizable logics. After this we will study a particular case of fibring of algebraizable logics: The case when the logics are defined by means of Hilbert-type axiomatization. We conclude this article by proving the existence of an isomorphism between the category of equivalent algebraic semantics and ALCO*. This opens the possibility to perform the fibring of algebraic semantics, sharing or not function symbols.

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1 Preliminaries

In this section we introduce the basic definitions which we propose to deal with: the category of propositional languages (which is also studied in [2]) and the category of algebraizable logics, as well as the categorial notion of fibring.

Definition 1.1 *(i) Fix a denumerable set* $\mathcal{V} = \{p_i\}_{i\in\mathbb{N}}$ *of* propositional variables*. A* propositional signature *is a family of pairwise disjoint sets* $C = \{C_k\}_{k\in\mathbb{N}}$ *such that* $|C| \cap \mathcal{V} = \emptyset$*, where* $|C| = \bigcup_{n\in\mathbb{N}} C_n$*; elements of* C_k *are called* connectives of arity k*. The* propositional language generated by C *(denoted by* $L(C)$*) is the free algebra (of words) generated by* $\mathcal{V}$ *over* C*.*
(ii) Given signatures C^1 *and* C^2*, a* signature morphism $h : C^1 \to C^2$ *is a map* $h : |C^1| \to L(C^2)$ *such that, if* $c \in C^1_n$ *then* $h(c) \in L(C^2)$ *such that* $h(c)$ *depends exactly on* $p_1, \ldots, p_n$*.*

If $h : C^1 \to C^2$ then there is exactly one extension $\widehat{h} : L(C^1) \to L(C^2)$ given by $\widehat{h}(p) = p$, if $p \in \mathcal{V}$, and $\widehat{h}(c(\beta_1, \ldots, \beta_k)) = h(c)(p_1/\widehat{h}(\beta_1), \ldots, p_k/\widehat{h}(\beta_k))$ if $c \in C^1_k$. By defining the composition of $C^1 \xrightarrow{h} C^2 \xrightarrow{k} C^3$ as $k \bullet h = \widehat{k} \circ h$, and putting $id_C : C \to C$ as $id_C(c) = c(p_1, \ldots, p_n)$ then we obtain a category of propositional languages, called PLAN.

Definition 1.2 *(i) The category ALCO of algebraizable logics is the category whose objects are consequence systems of the form* $\mathcal{L} = \langle C, \vdash \rangle$*, where* C *is a PLAN-object and* $\vdash \subseteq \wp(L(C)) \times L(C)$ *is a standard consequence relation (that is, extensive, transitive, monotonic, structural and finitary, cf. [10]), which is algebraizable in the sense of Blok-Pigozzi (see [1]). Given* $\mathcal{L}_i = \langle C^i, \vdash_i \rangle$ $(i = 1, 2)$*, an ALCO-morphism* $h : \mathcal{L}_1 \to \mathcal{L}_2$ *is a PLAN-morphism* $h : C^1 \to C^2$ *satisfying:*
(a) if $\Gamma \vdash_1 \alpha$*, then* $h(\Gamma) \vdash_2 h(\alpha)$*;*
(b) there are algebraizators $\langle (\epsilon^i, \delta^i), \Delta^i \rangle$ *for* $\mathcal{L}_i$ $(i = 1, 2)$*, respectively, such that, for every formula* ϕ, ψ *of* $L(C^1)$*, it holds* $h(\phi)\Delta^2 h(\psi) \vdash_2 h(\phi)h(\Delta^1)h(\psi)$*.*
Composition and identity maps are as in PLAN.
(ii) The category $ALCO^*$ *is the subcategory of ALCO such that the objects are the same, but the morphism are ALCO-morphisms satisfying:*
(c) there are algebraizators $\langle (\epsilon^i, \delta^i), \Delta^i \rangle$ *for* $\mathcal{L}_i$ $(i = 1, 2)$*, respectively, such that, for every formula* ϕ, ψ *of* $L(C^1)$*, it holds* $h(\phi)\Delta^2 h(\psi) \dashv\vdash_2 h(\phi)h(\Delta^1)h(\psi)$*.*[2]

In the definitions above, we can substitute "there are algebraizators" by "for every algebraizator", because of the properties of algebraizators.

The category ALCO* is not a full subcategory of ALCO, as shows the following example:

Example 1.3 Let us consider Sette's propositional paraconsistent logic P^1, defined in [9] over a signature C such that $|C| = \{\neg, \vee, \wedge, \Rightarrow\}$. In the positive fragments, P^1 coincide with classical logic CPC. In [7] it was proven that P^1 is

[2] $\Gamma_1 \vdash \Gamma_2$ means $\Gamma_1 \vdash \gamma$ for every $\gamma \in \Gamma_2$; and $\Gamma_1 \dashv\vdash \Gamma_2$ means $\Gamma_1 \vdash \Gamma_2$ and $\Gamma_2 \vdash \Gamma_1$.

algebraizable, where $\Delta_{P^1} = \{p_1 \Rightarrow p_2, p_2 \Rightarrow p_1, \neg p_1 \Rightarrow \neg p_2, \neg p_2 \Rightarrow \neg p_1\}$[3]. Now, let $P^1_{\Rightarrow}$ be the $\{\Rightarrow\}$-fragment of P^1. Since it coincides with the implicational fragment of CPC, it is algebraizable with $\Delta' = \{p_1 \Rightarrow p_2, p_2 \Rightarrow p_1\}$. It is clear that, considering the inclusion map $inc : P^1_{\Rightarrow} \to P^1$, $inc(\Delta')$ cannot be an equivalence set for P^1. Therefore, inc is an ALCO-morphism, but not an ALCO*-morphism.

We adapt the categorial notion of fibring introduced in [8], defining the following:

Definition 1.4 *Let $\mathcal{D}$ be a category of propositional logic system based on the category PLAN, that is, such that there is a forgetful functor $N : \mathcal{D} \to PLAN$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two $\mathcal{D}$-objects.*
(i) The unconstrained fibring *of $\mathcal{L}_1$ and $\mathcal{L}_2$ (denoted by $\mathcal{L}_1 \bigoplus \mathcal{L}_2$) is the coproduct in $\mathcal{D}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ (if it exists).*
(ii) Let $D = \{i_j : C \to N(\mathcal{L}_j)\}_{j=1,2}$ be diagram in PLAN formed by monomorphisms. Suppose that there exists the coproduct $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ in $\mathcal{D}$, and consider the coproduct $N(\mathcal{L}_1) \bigoplus N(\mathcal{L}_2) = N(\mathcal{L}_1 \bigoplus \mathcal{L}_2)$ of $N(\mathcal{L}_1), N(\mathcal{L}_2)$ in PLAN, with canonical injections $k_j : N(\mathcal{L}_j) \to N(\mathcal{L}_1 \bigoplus \mathcal{L}_2)$ $(j = 1, 2)$. The constrained fibring of $\mathcal{L}_1$ and $\mathcal{L}_2$ by sharing *D is the codomain $\mathcal{L}$ of the cocartesian lifting $q : \mathcal{L}_1 \bigoplus \mathcal{L}_2 \to \mathcal{L}$ of the coequalizer $q : N(\mathcal{L}_1 \bigoplus \mathcal{L}_2) \to C'$ of $\{k_1 \centerdot i_1, k_2 \centerdot i_2\}$ in PLAN (if the cocartesian lifting exists).*

2 Categorial Fibring in ALCO*

In this section we will prove that ALCO* has constrained and unconstrained fibring.

From [10] we know that the set $Stan(L(C))$ of standard consequence systems defined over the language $L(C)$, ordered by inclusion (of the consequence relations), is a complete lattice. Using this fact, we can define the following:

Definition 2.1 *Let $\mathcal{L}_i = \langle C^i, \vdash_i \rangle$ $(i = 1, 2)$ be two consequence systems in ALCO*. Consider the set $\mathcal{F}$ of standard consequence operators $\vdash$ over the coproduct $C = C^1 \bigoplus C^2$ such that:*
(i) $\Gamma \vdash_i \alpha$ implies $\Gamma \vdash \alpha$ for every $\Gamma \cup \{\alpha\} \subseteq L(C^i)$ $(i = 1, 2)$;
(ii) $\varphi \Delta^i \psi \vdash \varphi \Delta^j \psi$, for some Δ^i and Δ^j equivalence sets of the consequence systems $\mathcal{L}_i$ and $\mathcal{L}_j$ $(i, j \in \{1, 2\})$.[4]
We define the consequence system $\mathcal{L}_{\mathcal{F}} = \langle C, \vdash_{\mathcal{F}} \rangle$ such that $\vdash_{\mathcal{F}}$ is the infimum of the family $\mathcal{F}$ in $Stan(L(C))$.

Then we can prove the following:

Proposition 2.2 *Let $\mathcal{L}_i$ $(i = 1, 2)$, and $\mathcal{L}_{\mathcal{F}}$ as in Definition 2.1. Then $\mathcal{L}$ is an algebraizable consequence system.*

[3]And, implicitly, it was proven that $\{p_1 \Rightarrow p_2, p_2 \Rightarrow p_1\}$ is not an equivalence set for P^1.
[4]As before, it is equivalent to require "for every equivalence sets Δ^i and Δ^j".

Theorem 2.3 *Let* $\mathcal{L}_i$ *(*$i = 1, 2$*), and* $\mathcal{L}_{\mathcal{F}}$ *as above. Then* $\langle \mathcal{L}_{\mathcal{F}}, inc_1, inc_2 \rangle$ *is the* $ALCO^*$*-coproduct of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, where* $inc_i : \mathcal{L}_i \to \mathcal{L}_{\mathcal{F}}$ *is the canonical injection (*$i = 1, 2$*).*

With respect to constrained fibring, we have:

Theorem 2.4 *Let* $N : ALCO^* \to PLAN$ *the forgetful functor. Then,* N *is a cofibration.*

From theorems 2.3 and 2.4 we obtain the following:

Theorem 2.5 $ALCO^*$ *has both constrained and unconstrained fibring.*

3 Fibring Algebraizable Hilbert Systems

In this section we analyze the special case of algebraizable consequence systems generated by axioms and inference rules, that is, Hilbert-style systems. A first approach to this problem was outlined in [4].

Definition 3.1 *(i) Given a signature* C *and a fixed countable set* $\Xi = \{\xi_i\}_{i\in\mathbb{N}}$ *disjoint of* $\mathcal{V} \cup |C|$*, the* schematic propositional language relative to C*, denoted by* $L(C, \Xi)$*, is the free algebra generated by* $\mathcal{V} \cup \Xi$ *over* C*.*
(ii) A schematic inference rule *is a pair* $\langle \Upsilon, \beta \rangle$ *such that* $\Upsilon \cup \{\beta\}$ *is a finite subset of* $L(C, \Xi)$*.*
(iii) A Hilbert system *is a pair* $\mathcal{H} = \langle C, P \rangle$ *where* C *is a signature and* P *is a set of schematic inference rules*
(iv) A function $\sigma : \Xi \to L(C)$ *is called an* instantiation*. Any instantiation can be extended to a unique homomorphism* $\widehat{\sigma} : L(C, \Xi) \to L(C)$*.*
(v) Given a Hilbert system $\mathcal{H} = \langle C, P \rangle$*, the* consequence system $\mathcal{L}_{\mathcal{H}} = \langle C, \vdash_{\mathcal{H}} \rangle$ induced by $\mathcal{H}$ *is defined as follows:* $\Gamma \vdash_{\mathcal{H}} \alpha$ *iff there exists a finite sequence* $\phi_1, ..., \phi_m$ *(*$m \geq 1$*) in* $L(C)$ *such that* $\phi_m = \alpha$ *and, for every* $1 \leq i \leq m$*,*
(a) $\phi_i \in \Gamma$*, or*
(b) there is an instantiation σ *and a schematic inference rule* $\langle \Upsilon, \beta \rangle$ *such that* $\widehat{\sigma}(\beta) = \phi_i$ *and* $\widehat{\sigma}(\Upsilon) \subseteq \{\phi_1, ..., \phi_{i-1}\}$*.*

In order to define the fibring in ALCO of Hilbert system, we need to generalize the schema language, allowing symbols for schema connectives.

Definition 3.2 *(i) Given a signature* C*, fix a family of pairwise disjoint sets* $\Theta = \{\Theta_n\}_{n\in\mathbb{N}}$ *such that* $\Theta_n \cap (|C| \cup \mathcal{V} \cup \Xi) = \emptyset$*, for every* $n \in \mathbb{N}$*; elements in* Θ_n *are called* n-ary schema connectives*. The language* $L(C, \Theta; \Xi)$ *is the free algebra generated over* $\mathcal{V} \cup \Xi$ *by* $\{C_n \cup \Theta_n\}_{n\in\mathbb{N}}$*.*
(ii) A realization *is a partial function* $\rho : \bigcup_{n\in\mathbb{N}} \Theta_n \to |C|$ *such that, if* $C_n \neq \emptyset$*, then* $\rho(\kappa) \in C_n$ *for every* $\kappa \in \Theta_n$*.*
(iii) Given an instantiation $\sigma : \Xi \to L(C)$ *and a realization* $\rho : \bigcup_{n\in\mathbb{N}} \Theta_n \to |C|$*, the* extended instantiation induced by ρ and σ *is the function* $\chi : L(C, \Theta; \Xi) \to L(C)$ *defined as follows:*

(a) $\chi(p) = p$ *for every* $p \in \mathcal{V}$*;*
(b) $\chi(\xi) = \sigma(\xi)$ *for every* $\xi \in \Xi$*;*
(c) $\chi(\kappa(\phi_1, \ldots, \phi_n)) = \rho(\kappa)(\chi(\phi_1), \ldots, \chi(\phi_n))$*, if* $\kappa \in \Theta_n$ *and* $\rho(\kappa)$ *is defined;*
(d) $\chi(\kappa(\phi_1, \ldots, \phi_n)) = p_0$*, if* $\rho(\kappa)$ *is not defined.*
(iv) An extended Hilbert-system *is a pair* $\mathcal{H} = \langle C, P \rangle$ *such that* C *is a signature and* P *is a set of inference rules as in Definition 3.1(ii), but now using* $L(C, \Theta; \Xi)$.
(v) An extended Hilbert-system $\mathcal{H}$ *induces a consequence system* $\mathcal{L}_{\mathcal{H}} = \langle C, \vdash_{\mathcal{H}} \rangle$ *as in Definition 3.1(v), but now using* $L(C, \Theta; \Xi)$ *and extended instantiations.*

Definition 3.3 *(i) The category ALHI*$^-$ *is the full subcategory of ALCO whose objects are consequence systems* $\mathcal{L} = \langle C, \vdash \rangle$ *of ALCO such that* $\vdash$ *is induced by some Hilbert system* $\mathcal{H}$*.*
(ii) The category ALHI is the full subcategory of ALCO whose objects are consequence systems $\mathcal{L} = \langle C, \vdash \rangle$ *of ALCO such that* $\vdash$ *is induced by some extended Hilbert system* $\mathcal{H}$*.*

In order to obtain a sufficient condition for the existence of unconstrained fibring in ALHI, we recall the following result, due to [1]:

Proposition 3.4 *Suppose that a standard consequence system* $\mathcal{L} = \langle C, \vdash \rangle$ *has a set* $\Delta = \{\Delta_i(p_1, p_2)\}_{i=1}^n$ *of formulas such that, for every formula* φ*,* ψ*:*
(i) $\vdash \varphi \Delta \varphi$*;*
(ii) $\varphi \Delta \psi \vdash \psi \Delta \varphi$*;*
(iii) $\varphi \Delta \psi,\ \psi \Delta \alpha \vdash \varphi \Delta \alpha$*;*
(iv) For every connective $c \in C_k$*, and every formula* $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k$*:*
$\varphi_1 \Delta \psi_1, \ldots, \varphi_k \Delta \psi_k \vdash c(\varphi_1, \ldots, \varphi_k) \Delta c(\psi_1, \ldots, \psi_k)$*;*
(v) $\varphi, \varphi \Delta \psi \vdash \psi$*;*
(vi) $\varphi, \psi \vdash \varphi \Delta \psi$*.*
Then $\mathcal{L}$ *is algebraizable.*

Theorem 3.5 *If* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are logics in ALHI satisfying the conditions of Proposition 3.4, then there exists the unconstrained fibring (i.e., the coproduct)* $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ *of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *in ALHI.*

With respect to ALHI$^-$, we also obtain a sufficient condition for the existence of fibring.

Definition 3.6 *: Let* $\mathcal{L} = \langle C, \vdash \rangle$ *be an algebraizable consequence system, and let* $\langle (\delta, \epsilon), \Delta \rangle$ *be an algebraizator of* $\mathcal{L}$*. We say that* $\mathcal{L}$ *is* equivalence-expressing *if there is a formula* $(p_1 \Leftrightarrow p_2) \in L(C)$ *such that, for every* $\phi, \psi \in L(C)$*:* $\phi \Delta \psi \dashv\vdash \phi \Leftrightarrow \psi$*.*

Proposition 3.7 *Let* $\mathcal{L}_i$ *(*$i = 1, 2$*) be two ALHI*$^-$*-objects, which are equivalence-expressing. Then, there exists the constrained fibring of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *by sharing* $\Leftrightarrow$*, where* $\{\Leftrightarrow\}$ *is the identification of* $\Leftrightarrow_1$ *and* $\Leftrightarrow_2$*.*

4 The Category $ASEM$ of Equivalent Algebraic Semantics

Based on [6], now we define the category of classes of algebras that are equivalent algebraic semantics for some algebraizable consequence system. We will use languages in PLAN to represent terms of quasivarieties.

Definition 4.1 *(i) Let K be a quasivariety and let $\models_K$ be the associated consequence relation (cf. [1]). Given sets of terms $\delta = \{\delta_i(x)\}_{i\leq n}$, $\varepsilon = \{\varepsilon_i(x)\}_{i\leq n}$ and $\Delta = \{\Delta_j(x,y)\}_{j\leq m}$, we say that $[(\delta,\varepsilon),\Delta]$ is a* deductivizator *of K if it verifies: $x \approx y \dashv\models_K \varepsilon(x\Delta y) \approx \delta(x\Delta y)$.*
(ii) Given a quasivariety K such that K has a deductivizator, we define a equivalence relation $\simeq_K$ between deductivizators as follows:
$[(\delta,\varepsilon),\Delta] \simeq_K [(\delta',\varepsilon'),\Delta']$ iff $\varepsilon(x) \approx \delta(x) \dashv\models_K \varepsilon'(x) \approx \delta'(x)$. The equivalence class of $[(\delta,\varepsilon),\Delta]$ under $\simeq_K$ will be denoted by $[\varepsilon,\delta,\Delta]_K$.

Definition 4.2 *The category ASEM of equivalent algebraic semantics is the category whose objects are triples $\mathcal{A} = \langle C, K, [\varepsilon,\delta,\Delta]_K\rangle$, where C is a signature, K a quasivariety which have a deductivizator, and $[\varepsilon,\delta,\Delta]_K$ is the equivalence class under $\simeq_K$ of some deductivizator $[(\delta,\varepsilon),\Delta]$ of K. Given objects $\mathcal{A}_i = \langle C^i, K_i, [\varepsilon^i,\delta^i,\Delta^i]_{K_i}\rangle$ $(i = 1,2)$, a morphism $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a PLAN-morphism $h : C^1 \to C^2$ satisfying the following:*
(a) If $\Gamma \models_{K_1} (\nu \approx \eta)$ then $h(\Gamma) \models_{K_2} h(\nu \approx \eta)$, for every set $\Gamma \cup \{\nu \approx \eta\}$ of K_1-equations;
(b) For every $\tau \in L(C^1)$, $\varepsilon^2(h(\tau)) \approx \delta^2(h(\tau)) \dashv\models_{K_2} h(\varepsilon^1(\tau)) \approx h(\delta^1(\tau))$.[5]

The fundamental result of this section is the following:

Theorem 4.3 *ALCO* and ASEM are isomophic categories.*

As a corollary we obtain the following:

Corollary 4.4 *ASEM has coproducts, and the functor $N : ASEM \to PLAN$ which associates each equivalent algebraic semantics with its underlying signature is a cofibration.*

From the previous corollary, we can obtain new equivalent algebraic semantics (sharing or not function symbols) from previous ones. This result suggest the implicit notion of "fibring of equational languages", modifying the definitions of the previous sections.

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