Finite algebraizability via possible-translations semantics

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Abstract

The general idea of combining logics also involves the concept of breaking logics into families of logics with lower semantical complexity. The possible-translations semantics (**PTS**'s) are a particularly apt tool for analyzing and providing semantical meaning and algebraic contents to certain complex logics as paraconsistent logics. This paper characterizes **PTS**'s and the concept of algebraizability via **PTS**'s in categorial terms, extending the concept of finitely-algebraizable (or Blok-Pigozzi algebraizable) logics.

Keywords: possible-translations semantics; finitely algebraizable logics; categories of logical systems.

Introduction

In a broad view, combining logics includes, besides synthesizing logical systems by means of compositional procedures (as for instance fibring), also the direction of analysing a logic in terms of less complex logics. Such a reversing procedure is called *splitting*, as opposite to the process of *splicing* (cf. [6]).

In this paper we propose a categorial characterization of the process of splitting logics called *possible-translation semantics* (**PTS**'s, cf. [5]). Such semantics are adequate to providing interpretation to several non-standard logics (as to paraconsistent and to many-valued logics, for instance).

On the other hand, by combining algebraic techniques with **PTS**'s one obtains a new notion of algebraizability (cf. [3]) extending the method of finitely algebraizable logics due to W. Blok and D. Pigozzi in [1]. This extended notion of algebraization offers a solution to the question of obtaining an exact algebraic counterpart to certain logics which are not amenable to the method of Blok and Pigozzi, as it is the case of various paraconsistent logics.

The main results of this paper are the characterization of the possible-translations semantics (**PTS**) and the concept of algebraizability via **PTS**'s in categorial terms, and the proof of the Finiteness Preservation Lemma (3.4): it is proven that the product of finitely-algebraizable propositional logics is also finitely-algebraizable, under certain conditions.

This is achieved by defining the categories **PS** of propositional languages, and **CR** of propositional logics (defined through consequence relations), and showing that they are closed under arbitrary products. This permits to specify the category **ACR** of algebraizable logics as a subcategory of **CR** (cf. [10]). Examples and some research directions are discussed.

1 Propositional languages

Definition 1.1 *A signature is a denumerable family* $\Sigma = \{\Sigma_k\}_{k\in\omega}$*, where each* Σ_k *is a set (of* connectives of arity k*) such that* $\Sigma_k \cap \Sigma_n = \emptyset$ *whenever* $k \neq n$*. The* domain *of* Σ *is the set* $|\Sigma| = \bigcup_{n\in\omega} \Sigma_n$*. We fix a denumerable set* $\mathcal{V} = \{p_k \ : \ k \in \omega, k \geq 1\}$ *of* (propositional) variables *such that* $p_k \neq p_n$ *whenever* $k \neq n$*. The* (propositional) language generated by Σ*, denoted by* $L(\Sigma)$*, is the algebra of type* Σ *freely generated by* $\mathcal{V}$*. Elements of* $L(\Sigma)$ *are called* formulas*. For every* $n \geq 0$ *consider the following sets:*

$$L(\Sigma)[n] = \{\varphi \in L(\Sigma) \ : \ \textit{the variables occurring in } \varphi \textit{ are exactly } p_1, \ldots, p_n\},$$

$$L(\Sigma)(n) = \{\varphi \in L(\Sigma) \ : \ \textit{the variables occurring in } \varphi \textit{ are among } p_1, \ldots, p_n\}.$$

Definition 1.2 *Let* Σ *be a signature. A substitution on* $L(\Sigma)$ *is a function* $\sigma{:}\mathcal{V} \to L(\Sigma)$*. We denote by* $\widehat{\sigma}$ *the unique extension of* σ *to an endomorphism* $\widehat{\sigma} : L(\Sigma) \to L(\Sigma)$*.*

Given $\varphi \in L(\Sigma)(n)$ and σ such that $\sigma(p_i) = \alpha_i$ $(i = 1, \ldots, n)$ then $\widehat{\sigma}(\varphi)$ will be denoted by $\varphi(\alpha_1, \ldots, \alpha_n)$.

Definition 1.3 *Let* Σ *and* Σ' *be signatures. A* signature morphism f *from* Σ *to* Σ'*, denoted* $\Sigma \xrightarrow{f} \Sigma'$*, is a map* $f : |\Sigma| \to L(\Sigma')$ *such that, if* $c \in \Sigma_n$ *then* $f(c) \in L(\Sigma')[n]$*.*

Given a signature morphism $\Sigma \xrightarrow{f} \Sigma'$, a map $\widehat{f} : L(\Sigma) \to L(\Sigma')$ can be defined in a natural way:

1. $\widehat{f}(p) = p$ if $p \in \mathcal{V}$;
2. $\widehat{f}(c) = f(c)$ if $c \in \Sigma_0$;
3. $\widehat{f}(c(\alpha_1, \ldots, \alpha_n)) = f(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n))$ if $c \in \Sigma_n$ and $\alpha_1, \ldots, \alpha_n \in L(\Sigma)$.

The extension $\widehat{f}$ of f is unique: if f, f' are signature morphisms such that $\widehat{f} = \widehat{f'}$ then $f = f'$. Moreover, the propositional variables occurring in φ and in $\widehat{f}(\varphi)$ are the same.

Definition 1.4 *Let* $\Sigma \xrightarrow{f} \Sigma'$ *and* $\Sigma' \xrightarrow{g} \Sigma''$ *be signature morphisms. The* composition $g \centerdot f$ *of* f *and* g *is the signature morphism* $\Sigma \xrightarrow{g \centerdot f} \Sigma''$ *given by the map* $\widehat{g} \circ f : |\Sigma| \to L(\Sigma'')$.

Definition 1.5 *The Category* **PS** *of (Propositional) languages is defined as follows:*

- *Objects: Propositional signatures (cf. Definition 1.1);*
- *Morphisms: Signature morphisms (cf. Definition 1.3);*
- *Composition: As in Definition 1.4;*
- *Identity morphisms: For every signature* Σ *the identity morphism* $\Sigma \xrightarrow{id_\Sigma} \Sigma$ *is defined by* $id_\Sigma(c) = c$ *(for* $c \in \Sigma_0$*) and* $id_\Sigma(c) = c(p_1, \ldots, p_n)$ *(for* $c \in \Sigma_n$*,* $n \geq 1$*).*

Proposition 1.6 **PS** *is s category with arbitrary (small) products.*

The proof that **PS** a category is just a verification, as usual. **PS** is proven to have arbitrary (small) products by defining appropriate terminal signatures and using previous definitions.

2 Consequence relations

In this section we introduce the category **CR** of (propositional) logics defined through consequence relations.

Definition 2.1 *A* (propositional) logic *is a pair* $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$*, where* Σ *is a signature (cf. Definition 1.1) and* $\vdash_{\mathcal{L}}$ *is a subset of* $\wp(L(\Sigma)) \times L(\Sigma)$ *satisfying the following properties, for every* $\Gamma \cup \Theta \cup \{\varphi\} \subseteq L(\Sigma)$*:*

- *If* $\varphi \in \Gamma$ *then* $\Gamma \vdash_{\mathcal{L}} \varphi$ *(Extensivity);*
- *If* $\Gamma \vdash_{\mathcal{L}} \varphi$ *and* $\Gamma \subseteq \Theta$ *then* $\Theta \vdash_{\mathcal{L}} \varphi$ *(Monotonicity);*
- *If* $\Gamma \vdash_{\mathcal{L}} \varphi$ *and* $\Theta \vdash_{\mathcal{L}} \psi$ *for all* $\psi \in \Gamma$ *then* $\Theta \vdash_{\mathcal{L}} \varphi$ *(Transitivity);*
- *If* $\Gamma \vdash_{\mathcal{L}} \varphi$ *then* $\Delta \vdash_{\mathcal{L}} \varphi$ *for some finite* $\Delta \subseteq \Gamma$ *(Finitariness);*
- *If* $\Gamma \vdash_{\mathcal{L}} \varphi$ *then* $\widehat{\sigma}(\Gamma) \vdash_{\mathcal{L}} \widehat{\sigma}(\varphi)$ *for every substitution* σ *(Structurality).*

Definition 2.2 *Let* $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ *and* $\mathcal{L}' = \langle \Sigma', \vdash_{\mathcal{L}'} \rangle$ *be logics. A* morphism of logics $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ *from* $\mathcal{L}$ *to* $\mathcal{L}'$ *is a* **PS***-morphism* $\Sigma \xrightarrow{f} \Sigma'$ *which is a* translation*, that is, it satisfies, for every* $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$*:*

$$\Gamma \vdash_{\mathcal{L}} \varphi \quad \textit{implies} \quad \widehat{f}(\Gamma) \vdash_{\mathcal{L}'} \widehat{f}(\varphi).$$

By defining composition of morphisms and identity morphisms as in **PS** we then obtain a category of (propositional) logics defined through consequence relations, called **CR**. A fundamental property of **CR** is the following:

Proposition 2.3 *The category* **CR** *has arbitrary (small) products.*

Proof: Let $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ be a family of logics, where I is a set and $\mathcal{L}_i = \langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$ for every $i \in I$. If $I = \emptyset$ then, taking the terminal signature $S^{\mathbf{1}}$, $L^{\mathbf{1}} = \langle S^{\mathbf{1}}, \wp(L(S^{\mathbf{1}})) \times L(S^{\mathbf{1}}) \rangle$ is a terminal object in **CR** being, therefore, the product of $\mathcal{F}$; given a logic $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ then the unique **PS**-morphism $\Sigma \xrightarrow{!} S^{\mathbf{1}}$ defines the unique morphism $\mathcal{L} \xrightarrow{!} L^{\mathbf{1}}$ in **CR**. If $I \neq \emptyset$ consider the product $\langle \Sigma^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ of $\{\Sigma^i\}_{i \in I}$ in **PS** (cf. Proposition 1.6). Define a relation $\vdash_{\mathcal{F}} \subseteq \wp(L(\Sigma^{\mathcal{F}})) \times L(\Sigma^{\mathcal{F}})$ as follows: $\Gamma \vdash_{\mathcal{F}} \varphi$ iff there exists a finite set $\Delta \subseteq \Gamma$ such that $\widehat{\pi}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$ for every $i \in I$. The rest of the proof consists in showing that $\mathcal{L}^{\mathcal{F}} = \langle \Sigma^{\mathcal{F}}, \vdash_{\mathcal{F}} \rangle$ is a logic and is not detailed here. $\square$

3 Products of algebraizable logics

In this section we prove that, given a (small) family $\mathcal{F}$ of finitely algebraizable logics (in the sense of [1]) satisfying a finite bounding condition, then the product of $\mathcal{F}$ in **CR** is also an algebraizable logic. This will be used in Section 5.

We begin by briefly recalling the basic definitions of [1].

Definition 3.1 *A logic* $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ *is* algebraizable *(in the sense of Blok-Pigozzi) if there exists a finite set* $\Delta = \{\Delta^i \ : \ 1 \leq i \leq n\}$ *of formulas in* $L(\Sigma)[2]$*, and a finite set* $\langle \varepsilon, \delta \rangle = \{\langle \varepsilon^i, \delta^i \rangle \ : \ 1 \leq i \leq m\}$ *contained in* $L(\Sigma)[1] \times L(\Sigma)[1]$ *such that, for every* $\varphi, \psi, \alpha \in L_\Sigma$*:*

1. $\vdash_{\mathcal{L}} \varphi \Delta \varphi$;
2. $\varphi \Delta \psi \vdash_{\mathcal{L}} \psi \Delta \varphi$;
3. $\varphi \Delta \psi,\ \psi \Delta \alpha \vdash_{\mathcal{L}} \varphi \Delta \alpha$;
4. $\varphi_1 \Delta \psi_1, \ldots, \varphi_k \Delta \psi_k \vdash_{\mathcal{L}} c(\varphi_1, \ldots, \varphi_k) \Delta c(\psi_1, \ldots, \psi_k)$ *for every* $c \in \Sigma_k$ *and every* $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k$ *in* $L(\Sigma)$;
5. $\varphi \vdash_{\mathcal{L}} \varepsilon(\varphi) \Delta \delta(\varphi)$, *and* $\varepsilon(\varphi) \Delta \delta(\varphi) \vdash_{\mathcal{L}} \varphi$.

We say that $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ *is an* algebraizator *for* $\mathcal{L}$.

Some remarks on the notation adopted in Definition 3.1. For any $\varphi, \psi \in L(\Sigma)$ then $\varphi \Delta \psi$ denotes the set of formulas $\{\Delta^i(\varphi, \psi) \ : \ 1 \leq i \leq n\}$, and $\varepsilon(\varphi) \Delta \delta(\varphi)$ denotes the set $\{\Delta^j(\varepsilon^i(\varphi), \delta^i(\varphi)) : \ 1 \leq j \leq n \text{ and } 1 \leq i \leq m\}$. And given sets Γ, Θ of formulas then $\Gamma \vdash_{\mathcal{L}} \Theta$ means that $\Gamma \vdash_{\mathcal{L}} \varphi$ for every $\varphi \in \Theta$. Following [10] we define the category **ACR** of algebraizable logics.

Definition 3.2 *The category* **ACR** *of* algebraizable logics *is the subcategory of* **CR** *defined as follows:*

- *Objects: logics* $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ *which are algebraizable (cf. Definition 3.1);*
- *Morphisms: a morphism* $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ *is a* **CR***-morphism* $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ *such that, if* $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ *and* $\langle \Delta', \langle \varepsilon', \delta' \rangle \rangle$ *are algebraizators for* $\mathcal{L}$ *and* $\mathcal{L}'$*, respectively, then* $p_1 \widehat{f}(\Delta) p_2 \vdash_{\mathcal{L}'} p_1 \Delta' p_2$ *and* $p_1 \Delta' p_2 \vdash_{\mathcal{L}'} p_1 \widehat{f}(\Delta) p_2$*, where* $p_1 \widehat{f}(\Delta) p_2$ *denotes the set of formulas* $\{\widehat{f}(\Delta^i)(p_1, p_2) \ : \ 1 \leq i \leq n\}$*;*
- *Composition and identity morphisms: inherited from* **CR**.

Remark 3.3 *From [1] we get the following: let* $\langle \Delta_1, \langle \varepsilon_1, \delta_1 \rangle \rangle$ *and* $\langle \Delta_2, \langle \varepsilon_2, \delta_2 \rangle \rangle$ *be two algebraizators for a logic* $\mathcal{L}$*. Then* $p_1 \Delta_2 p_2 \vdash_{\mathcal{L}} p_1 \Delta_1 p_2$ *and* $p_1 \Delta_1 p_2 \vdash_{\mathcal{L}} p_1 \Delta_2 p_2$*. Therefore, a* **CR***-morphism* $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ *is a* **ACR***-morphism iff there are algebraizators* $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ *and* $\langle \Delta', \langle \varepsilon', \delta' \rangle \rangle$ *for* $\mathcal{L}$ *and* $\mathcal{L}'$*, respectively, such that* $p_1 \widehat{f}(\Delta) p_2 \vdash_{\mathcal{L}'} p_1 \Delta' p_2$ *and* $p_1 \Delta' p_2 \vdash_{\mathcal{L}'} p_1 \widehat{f}(\Delta) p_2$*. On the other hand, from [10] we have the following result: a* **CR***-morphism* $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ *is a* **ACR***-morphism iff, for every algebraizator* $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ *for* $\mathcal{L}$*, the pair* $\langle \widehat{f}(\Delta), \langle \widehat{f}(\varepsilon), \widehat{f}(\delta) \rangle \rangle$ *is an algebrizator for* $\mathcal{L}'$*.*

Now we will prove that the product of a (small) family of algebraizable logics satisfying a finitely bounding condition is algebraizable.

Theorem 3.4 *(Finiteness Preservation) Let* $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ *be a family of algebraizable logics, where* I *is a set and* $\mathcal{L}_i = \langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$ *for every* $i \in I$*. Assume that* $\mathcal{F}$ *has the following property: there are natural numbers* n *and* m *such that, for every* $i \in I$*, there is an algebraizator* $\langle \Delta_i, \langle \varepsilon_i, \delta_i \rangle \rangle$ *for* $\mathcal{L}_i$ *such that* Δ_i *has at most* n *elements, and* $\langle \varepsilon_i, \delta_i \rangle$ *has at most* m *elements. Then, there exists the product in* **ACR** *of* $\mathcal{F}$*.*

Proof: By hypothesis we can take, for any $i \in I$, finite sequences

$$\Delta_i^1 \cdots \Delta_i^n$$

$$\langle \varepsilon_i^1, \delta_i^1 \rangle \cdots \langle \varepsilon_i^m, \delta_i^m \rangle$$

such that $\langle \Delta_i, \langle \varepsilon_i, \delta_i \rangle \rangle$ is an algebraizator for $\mathcal{L}_i$, where

$$\Delta_i = \{\Delta_i^1, \ldots, \Delta_i^n\}$$

and $\langle \varepsilon_i, \delta_i \rangle = \{\langle \varepsilon_i^1, \delta_i^1 \rangle, \ldots, \langle \varepsilon_i^m, \delta_i^m \rangle\}$, for every $i \in I$. In fact, it is enough to take, for every $i \in I$, an algebraizator with at most n elements in Δ_i and at most m elements in $\langle \varepsilon_i, \delta_i \rangle$ and list their elements, repeating, if necessary, some elements, in order to define sequences of length n and m, respectively.

Now, consider the product $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ in **CR** of family $\mathcal{F}$, and define the following formulas in $L(\Sigma^{\mathcal{F}})$:

- $\Delta^j = (\Delta_i^j)_{i\in I}(p_1, p_2)$ for $1 \leq j \leq n$;
- $\varepsilon^j = (\varepsilon_i^j)_{i\in I}(p_1)$ for $1 \leq j \leq m$;
- $\delta^j = (\delta_i^j)_{i\in I}(p_1)$ for $1 \leq j \leq m$.

Finally, let $\Delta = \{\Delta^i \;:\; 1 \leq i \leq n\}$ and $\langle\varepsilon, \delta\rangle = \{\langle\varepsilon^i, \delta^i\rangle \;:\; 1 \leq i \leq m\}$. It can be proven that $\langle\Delta, \langle\varepsilon, \delta\rangle\rangle$ is an algebraizator for $\mathcal{L}^{\mathcal{F}}$ (cf. Definition 3.1). The rest of the proof consists of proving (by induction on the length of formulas) that the clauses of Definition 3.1 are satisfied. □

4 Possible-translation semantics

Besides considering synthesis of given logics by means of a combination process (as, for instance, fibring) in order to obtain a new logic, it is also convenient to be able to split a logic into a family of simpler logics. This kind of 'reverse' technique is what was called *splitting logics*, as opposite to the process of *splicing logics* (cf. [6]). In this section we provide a categorial characterization of the process of splitting logics called *possible-translation semantics* (cf. [5]).

We begin by adapting the original definitions of [5] to our formalism.

Definition 4.1 *Let $\mathcal{L} = \langle\Sigma, \vdash_{\mathcal{L}}\rangle$ be a logic, and let $\{\mathcal{L}_i\}_{i\in I}$ be a family of logics such that I is a set and $\mathcal{L}_i = \langle\Sigma^i, \vdash_{\mathcal{L}_i}\rangle$ for every $i \in I$. Let $\mathcal{L} \xrightarrow{f_i} \mathcal{L}^i$ be a* **CR***-morphism for every $i \in I$. Then $P = \langle\{\mathcal{L}_i\}_{i\in I}, \{f_i\}_{i\in I}\rangle$ is a* possible-translation *for $\mathcal{L}$ (in short, a* **PTS***) if, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$,*

$$\Gamma \vdash_{\mathcal{L}} \varphi \;\text{ iff there is a finite } \Delta \subseteq \Gamma \text{ such that } \widehat{f}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{f}_i(\varphi) \;\text{ for every } i \in I.$$

The meaning of having a **PTS** for a logic $\mathcal{L}$ is that $\mathcal{L}$ splits into the family $\{\mathcal{L}_i\}_{i\in I}$ through the translations $\{f_i\}_{i\in I}$.

Inspired by [8] we say that a **CR**-morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is a *conservative translation* if, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \;\text{ iff }\; \widehat{f}(\Gamma) \vdash_{\mathcal{L}'} \widehat{f}(\varphi).$$

Using the results stated in the previous sections we characterize **PTS**'s in categorial terms.

We also show below that **PTS**'s for a logic induce conservative translations over products of families of logics in **CR**, and vice-versa. Such (induced) conservative translations turn out to be apt to extend the method of finite algebraizability with interesting applications, as shown in Section 5.

Theorem 4.2 *Given a possible-translation semantics for a logic $\mathcal{L}$ there exists a conservative translations $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, (where $\mathcal{L}'$ is a product in* **CR** *of some family of logics), and vice-versa.*

Proof: Fix a logic $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$. If P is a **PTS** for $\mathcal{L}$ then it is possible to define a conservative translation $\mathcal{L} \stackrel{\mathbf{t}(P)}{\rightarrow} \mathbf{L}(P)$, where $\mathbf{L}(P)$ is a product in **CR** of some (small) family of logics, encoding P. Conversely, given a conservative translation $\mathcal{L} \stackrel{f}{\rightarrow} \mathcal{L}'$, where $\mathcal{L}'$ is a product of logics, we can define a **PTS** for $\mathcal{L}$ called **PTS**(f) encoding f such that these assignments (**t** and **PTS**) are one inverse of the other. Calculations are omitted here. □

5 Algebraizing logics by possible-translations semantics

The results above give support to a method for algebraizing logics using **PTS**'s which extends the well-known method of finite algebraizability of [1]. The idea, introduced in [3], is the following: consider a logic $\mathcal{L}$, and let $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ be a **PTS** for $\mathcal{L}$. Suppose that every $\mathcal{L}_i$ is algebraizable, and assume that the family $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ satisfies the condition of Theorem 3.4. Then, by Theorem 4.2, the product $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ of family $\mathcal{F}$ in **ACR** encodes P. Moreover, it is possible to build an algebraizator for $\mathcal{L}^{\mathcal{F}}$ from a bounded (in the sense of Theorem 3.4) family of algebraizators for $\mathcal{F}$. This shows that there exists a conservative translation $\mathcal{L} \stackrel{f}{\rightarrow} \mathcal{L}^{\mathcal{F}}$, where $\mathcal{L}^{\mathcal{F}}$ is an algebraizable logic. The conservative translation f is a link between $\mathcal{L}$ and $\mathcal{L}^{\mathcal{F}}$ that preserves derivability (see comments after Definition 4.1) and so, using the algebraization for $\mathcal{L}^{\mathcal{F}}$, one obtains a kind of 'remote' algebraization for $\mathcal{L}$: in order to algebraically analyze $\mathcal{L}$, it is sufficient to translate $\mathcal{L}$ into $\mathcal{L}^{\mathcal{F}}$ and then analyze the result using the algebraic resources of $\mathcal{L}^{\mathcal{F}}$.

Here a concrete example is considered. Possible-translations semantics are used (cf. [5] and [12]) to obtain new semantics for the paraconsistent systems C_n introduced in [7]. Such semantics permit to characterize the logics C_n in terms of a family $\mathcal{F}_n$ of three-valued logics **LFI1** (cf.[4]), which are equivalent to the three-valued paraconsistent logic J_3 (introduced in [9]).

It turns out that J_3, **LFI1** and the three-valued Łukasiewicz logic $Ł_3$ are all finitely algebraizable with the same equivalent quasivariety, to wit, the quasivariety of the three-valued Moisil algebras, as shown in [2], p. 43.

It is clear then that logics J_3 (or **LFI1**) in the family $\mathcal{F}_n$ are finitely algebraizable and their algebraization satisfy the conditions of Finiteness Preservation Theorem 3.4. As a consequence, the product $\mathcal{L}^{\mathcal{F}_n}$ of the family $\mathcal{F}_n$ is also algebraizable, as argued in [3].

Furthermore, as a consequence of Theorem 4.2 there exists a conservative translations from each C_n into $\mathcal{L}^{\mathcal{F}_n}$.

This shows that our categorial characterization extends the concept of finite algebraizability in adequate terms, offering a non *ad hoc* solution to the question of algebraizing logics in general, as it amply extends the method of [1]. Other interesting questions, as the characterization of logics having the Craig interpolation property for consequence relation, can be recast here as a challenging problem: indeed, it is known (see [11] page 43 for a discussion) that a logic enjoying the deduction-detachment theorem has the Craig interpolation property iff its algebraization has the amalgamation property. Since the amalgamation property can be seen as a universal construction in the (sub)category **CR** of algebraizable logics, it remains to know whether this would correspond to any form of Craig interpolation property.

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